Continuous-time Equilibrium Returns in Markets with Price Impact and Transaction Costs

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Motivation

Several financial markets are thin

Undeniable facts:

- In many contemporaneous financial markets, there are large institutional investors possessing market power ⇒ producing price impact.
- In such non-competitive markets exogenous transaction costs are normally present.
- Both of these features are expected to affect the market in a non-standard manner.

In this work

We consider a continuous-time equilibrium model, with quadratic transaction costs on trading and

- strategic investors (*taking their price impact into account*)
- with mean-variance preferences (*risk averse*) and
- heterogeneous and exogenous random income (*different hedging needs*)

and we aim to endogenously determine the assets' expected returns.

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Contributions

Closed-form expressions of Nash equilibria and comparisons

- We apply a rather simple endogenous price-impact model, where investors' demands internalize their impact on assets' drift.
- W/o transaction costs, the Nash equilibrium drift is unique and nice!
- While the equilibrium models with transaction costs are generally intractable, under *common risk aversion*, we get the Nash equilibrium through a system of linear FBSDEs, which yields a closed-form expression for the equilibrium drift.
- We hence are able to compare the market's drift under different types of equilibria w/ and w/o transaction costs and w/ and w/o price impact.
- In short, investors' price impact increases the part of the equilibrium return that stems from transaction costs (when investors are more than two).
- Under common risk aversion and absence of noise traders, Nash equilibrium w/ and w/o costs coincide (a result that holds for competitive equilibria too).
- Investors with relatively low risk aversion get higher utility gains in Nash equilibria.

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Related literature

Linked with two strands of the literature: equilibrium models with price impact and the optimal investment and equilibrium pricing under transaction costs.

Non-competitive equilibrium models with price impact

- Exogenous price impact: Cuoco and Cvitanic'98, Almegren and Chriss '00, Huberman and Werner '04, Almegren et al. '05.
- Endogenous price impact: Vayanos '99, Vives '11, Rostek and Werekta '15, Malamud and Weretka '17, A. '17, Choi et al. '21.

Equilibrium models with transaction costs

- W/o price impact: Bouchard et al. '18, Herdegen et al. '21.
- Discrete time: Buss and Dumas '19.
- Deterministic asset prices: Vayanos and Vila '99.
- Exogenous price impact and transaction costs: Schied and Zhang '19, Luo and Scheid '19, Cordoni and F. Lillo '24.

Outline

- The market's setup.
- Price-impact equilibrium returns: Without transaction costs.
- Price-impact equilibrium returns: With transaction costs.
- Closing remarks.

The market and its participants

Assets

One trivial riskless asset and d risky assets with dynamics:

$$\frac{dS^i(t)}{S^i(t)} = dR^i(t), \quad S^i(0) > 0,$$

$$dR^i(t) = \nu^i(t)dt + \sum_{j=1}^{d} \sigma^{ij} dW^j(t), \quad R^i(0) = 0,$$

for $1 \leq i \leq d$ over a finite time horizon T.
✓ Drift processes $\nu^i(t)$'s are going to be endogenously derived.

Participants

→ N investors with mean-variance preferences and random endowments Y_m:

$$dY_m(t) = (\zeta_m(t))' \sigma dW(t) \qquad \text{Exogenous endowment}$$

$$dX_m(t) = (\phi_m(t))' dR(t) + dY_m(t) \qquad \text{Total wealth}$$

→ Exogenous liquidity providers/noise traders' demand $\psi(t)$.

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The frictionless market

The mean-variance objective functional

For any given ν, each investor imposes a dynamic mean-variance investment objective up to time T that takes into account her random endowment:

$$\mathbb{E}\left[\int_0^T e^{-rt}\left(\overbrace{(\phi_m(t))' dR(t) + dY_m(t)}^{Wealth} - \frac{1}{2\delta_m} d\overbrace{\left[\int_0^{\cdot} (\phi_m(s))' dR(s) + Y_m\right]}^{\text{“Variance” of Wealth}}(t)\right)\right] =$$

$$\mathbb{E}\left[\int_0^T e^{-rt}\left((\phi_m(t))'\nu(t) - \frac{1}{2\delta_m}(\phi_m(t) + \zeta_m(t))'\Sigma(\phi_m(t) + \zeta_m(t))\right)dt\right] \to \max.$$

- The frictionless with no price impact optimal investment for investor m:

$$\widehat{\phi}_m := \delta_m \Sigma^{-1}\nu - \zeta_m.$$

- What about ν?

The frictionless market, *cont'd*

- Recall the presence of noise traders' demand: ψ.

The equilibrium condition (market clearing)

$$\widehat{\phi}_1(t) + \cdots + \widehat{\phi}_N(t) + \psi(t) = 0.$$

The competitive equilibrium returns

$$\mu := \frac{\Sigma(\zeta - \psi)}{\delta},$$

where $\zeta := \sum_{m=1}^{N} \zeta_m$, $\delta := \sum_{m=1}^{N} \delta_m$.

Roughly,

- $\zeta \uparrow \implies \mu \uparrow$: Large aggregate exposure to market risk leads to decreased demand $\implies$ lower current price (i.e. higher future return).
- $\psi \uparrow \implies \mu \downarrow$: More exogenous demand $\implies$ higher current price (i.e. lower future return).

A model for price impact

Extracting price impact from equilibrium condition

Take the position of the first investor. Assuming that the rest of the investors act as price takers, her demand appears explicitly at the equilibrium returns:

$$\phi_1(t) + \sum_{m=2}^{N} \Big(\delta_m \Sigma^{-1} \nu(t) - \zeta_m(t) \Big) + \psi(t) = 0,$$

Hence, we may write equilibrium drift as a function of the first investor's demand:

$$\nu(t;\phi_1) = \frac{\Sigma(\zeta_{-1}(t) - \psi(t) - \phi_1(t))}{\delta_{-1}}.$$

- Her adjusted objective (which internalizes her impact) becomes:

$$\mathbb{E}\left[\int_0^T e^{-rt}\Big((\phi_1(t))' \nu(t,\phi_1) - \frac{1}{2\delta_1}(\phi_1(t) + \zeta_1(t))' \Sigma(\phi_1(t) + \zeta_1(t)) \Big) dt\right] \to \max.$$

- This gives her frictionless *best-response* strategy:

$$\widetilde{\phi}_1 := \frac{\lambda_1(\zeta_{-1} - \psi) - \lambda_{-1}\zeta_1}{\lambda_1 + 1} = \frac{\widehat{\phi}_1(\mu)}{\lambda_1 + 1},$$

where $\lambda_n := \delta_n/\delta$, $\lambda_{-n} := 1 - \lambda_n$, $\zeta_{-n} := \sum_{m=1, m\neq n}^{N} \zeta_m$, for each $1 \leq n \leq N$.

Towards Nash equilibrium

It is like revealing different risk exposure...

✓ We observe that the best-response strategy $\widetilde{\phi}_1$ reveals different exposure to market risk than ζ_1.

- Indeed, the strategic investor drives the market at a different equilibrium by revealing the risk exposure:

$$\tilde{\zeta}_1(t) = \frac{1}{1+\lambda_1}\zeta_1(t) + \frac{\lambda_1^2}{1-\lambda_1^2}(\zeta_{-1}(t) - \psi(t)).$$

Nash equilibrium without transaction costs

When all investors apply the same strategy $\Longrightarrow$ *A linear system in investors' demands*:

$$\text{(NaSy)} \qquad \breve{\phi}_m = \frac{\lambda_m(\zeta_{-m}(\breve{\phi}) - \psi) - \lambda_{-m}\zeta_m}{\lambda_m + 1}, \quad \text{for each } 1 \leq m \leq N.$$

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Frictionless Nash equilibrium

Proposition

The Nash equilibrium drift (following by the solution of system (NaSy)) is given by

$$\breve{\mu} := \frac{\Sigma}{\delta} \frac{(\zeta - \psi) - \sum_{n=1}^{N} \lambda_n \zeta_n}{1 - \sum_{n=1}^{N} \lambda_n^2}.$$

- Assume that investors have equal risk tolerances $\delta_m = \bar{\delta}$, for all m. Then:

$$\breve{\mu}(t) - \mu(t) = -\frac{\Sigma}{\bar{\delta}} \frac{1}{N(N-1)} \psi(t).$$

✓ Roughly, when $\psi > 0$ (resp. $\psi < 0$) $\Longrightarrow$ Price impact induces lower (resp. higher) required market return.

✓ When there is no noise traders $\Longrightarrow$ the equilibrium returns do not change due to price impact.

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The market with transaction costs

The objective with transaction costs but no price impact

$$\mathbb{E}\left[\int_0^T e^{-rt}\left((\phi_n(t))'\nu(t) - \frac{1}{2\delta_n}(\phi_n(t)+\zeta_n(t))'\Sigma(\phi_n(t)+\zeta_n(t)) - \underbrace{(\dot{\phi}_n(t))'\Lambda\dot{\phi}_n(t)}_{\text{Transaction costs}}\right)dt\right],$$

where

$$d\phi_m(t) = \dot{\phi}_m(t)dt, \quad 1 \le m \le N$$

✓ Microstructure of transaction costs: $\Delta\phi$ induces a purely temporary impact $\Lambda\Delta\phi$ (Garleanu and Pederson '16).

Characterization of m's optimal demand, Bouchard et al. '18

$$d\phi_n(t) = \dot{\phi}_n(t)dt, \quad \phi_n(0) = 0,$$

$$d\dot{\phi}_n(t) = dM_n(t) + \frac{\Lambda^{-1}\Sigma}{2\delta_n}(\phi_n(t) - \widehat{\phi}_n(t))dt + r\dot{\phi}_n(t)dt, \quad \dot{\phi}_n(T) = 0.$$

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Transaction costs and price impact

Challenges:

- The transaction costs complicate the equilibrium returns even w/o price impact.
- There is no explicit expression of the form of competitive equilibrium returns.

However, we may get the equilibrium formulas when

- Equal risk tolerances ($\delta_m = \bar{\delta}$, for all m) or
- $N = 2$.

Extracting price impact in the market with transaction costs

Take again the position of the first investor. Note that the market clearing condition holds for the rate of tradings too.

$$\phi_1(t) + \sum_{m=2}^{N} \phi_{\Lambda,m}(t;\nu) + \psi(t) = 0,$$

$$\dot{\phi}_1(t) + \sum_{m=2}^{N} \dot{\phi}_{\Lambda,m}(t;\nu) + \dot{\psi}(t) = 0.$$

- *How does price impact affect the optimization objective in this case?*

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- *How does price impact affect the optimization objective in this case?*

Transaction costs and price impact, *cont'd*

Let $\delta_m = \bar{\delta}$, for all m.

The strategic investor's objective with transaction costs and price impact becomes:

$$\mathbb{E}\Bigg[\int_0^T e^{-rt}\Bigg((\phi_1(t))'\Bigg(\overbrace{\frac{\Sigma(\zeta_{-1}(t)-\phi_1(t)-\psi(t))}{\bar{\delta}(N-1)}}^{\text{Frictionless impact}} + \text{Extra noise term}\Bigg)$$

$$-\frac{1}{2\bar{\delta}}(\phi_1(t)+\zeta_1(t))'\Sigma(\phi_1(t)+\zeta_1(t)) - (\dot{\phi}_1(t))'\Bigg(\Lambda + \underbrace{\frac{2\Lambda}{N-1}}_{\text{Cost due to price impact}}\Bigg)\dot{\phi}_1(t)\Bigg)dt\Bigg] \to \max.$$

Proposition: Characterization of optimal demand with price impact

Under some mild integrability conditions, the best-response under transaction costs solves

$$\text{(TCs)}\quad \begin{aligned} d\tilde{\phi}_{\Lambda,1}(t) &= \dot{\tilde{\phi}}_{\Lambda,1}(t)dt, \quad \tilde{\phi}_{\Lambda,1}(0) = 0, \\ d\dot{\tilde{\phi}}_{\Lambda,1}(t) &= d\tilde{M}_1(t) + \frac{\Lambda^{-1}\Sigma}{2\bar{\delta}}(\tilde{\phi}_{\Lambda,1}(t) - \mathbb{TP}_1(t))dt + r\dot{\tilde{\phi}}_{\Lambda,1}(t)dt, \quad \dot{\tilde{\phi}}_{\Lambda,1}(T) = 0, \end{aligned}$$

where:

$\mathbb{TP}_1$ is frictionless best-response + extra noise term.

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Solution

Equation (TCs) belongs to a well-studied class of linear FBSDEs with explicit solution:

Proposition

Under some mild integrability conditions, the unique optimal demand of a strategic investor admits the following explicit form:

$$\widetilde{\phi}_{\Lambda,1}(t) = \int_0^t e^{-\int_s^t F(u)du} \widetilde{\mathbb{TP}}_1(s)ds,$$

where

$$F(t) := -\left(\Delta G(t) - \frac{r}{2}\dot{G}(t)\right)^{-1} B\dot{G}(t), \quad \Delta := B + \frac{r^2}{4} I_d, \quad G(t) := \cosh(\sqrt{\Delta}(T-t))$$

$$\widetilde{\mathbb{TP}}_1(t) := \left(\Delta G(t) - \frac{r}{2}\dot{G}(t)\right)^{-1} \mathbb{E}\left[\int_t^T \left(\Delta G(s) - \frac{r}{2}\dot{G}(s)\right) B e^{-\frac{r}{2}(s-t)} \mathbb{TP}_1(s)ds \middle| \mathcal{F}(t)\right]$$

and $B = \Lambda^{-1}\Sigma/2\bar{\delta}$.

✓ *As in the competitive case, common risk tolerance and absence of noise traders implies no effect on equilibrium returns due to transaction costs!*

Nash equilibrium with transaction costs

Characterization of optimal under the price impact of all investors

Under some mild integrability conditions, when all investors are strategic the Nash equilibrium solves the following system

$$\text{(NasEq)} \qquad \begin{aligned} d\breve{\phi}_{\Lambda,m}(t) &= \dot{\breve{\phi}}_{\Lambda,m}(t)dt, \\ d\dot{\breve{\phi}}_{\Lambda,m}(t) &= d\widetilde{M}_m(t) + \frac{\Lambda^{-1}\Sigma}{2\bar{\delta}}(\breve{\phi}_{\Lambda,m}(t) - \breve{\mathbb{TP}}_m(t))dt + r\dot{\breve{\phi}}_{\Lambda,m}(t)dt, \end{aligned}$$

for each $1 \leq m \leq N$, where:

$\breve{\mathbb{TP}}_m = m$'s frictionless best-response for $\zeta_{-m} \to \zeta_{-m}(\breve{\phi}_\Lambda)$ + extra noise term.

Theorem

System (NasEq) admits a unique solution and

$$\breve{\mu}_\Lambda := \breve{\mu} + \frac{2\Lambda}{N(N-1)} \times \text{Extra noise term.}$$

where $\breve{\mu}$ is the corresponding frictionless Nash equilibrium (when $\delta_m = \bar{\delta}$).

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The case of $N = 2$ and corollaries

When strategic investors are only two, the common risk tolerance is not necessary.

Theorem

Let $N = 2$. In a market with transaction costs, the unique Nash equilibrium returns are given by:

$$\hat{\mu}_\Lambda := \breve{\mu} + \Lambda \times \text{ Extra noise term},$$

where $\breve{\mu}$ is the corresponding frictionless Nash equilibrium.

Corollaries

- Connection between Nash w/o TCs $\breve{\mu}$ and Competitive w/o TCs μ:
  ✓ Equal risk tolerances & no noise traders $\Longrightarrow$ $\breve{\mu}$ reverts to μ.
- Connection between Nash w/ TCs $\breve{\mu}_\Lambda$ and Nash w/o TCs $\breve{\mu}$:
  ✓ Equal risk tolerances (or $N = 2$) and as $\Lambda \downarrow \Longrightarrow \breve{\mu}_\Lambda$ reverts to $\breve{\mu}$.
  ✓ When $N > 2$, the effect of transaction costs on equilibrium increases due to price impact.
- Connection between Nash w/ TCs $\breve{\mu}_\Lambda$ and Competitive w/o TCs μ:
  ✓ Equal risk tolerances & no noise traders $\longrightarrow \breve{\mu}_\Lambda$ reverts to μ.

The case of $N = 2$ and corollaries

When strategic investors are only two, the common risk tolerance is not necessary.

Theorem

Let $N = 2$. In a market with transaction costs, the unique Nash equilibrium returns are given by:

$$\hat{\mu}_\Lambda := \breve{\mu} + \Lambda \times \text{ Extra noise term},$$

where $\breve{\mu}$ is the corresponding frictionless Nash equilibrium.

Corollaries

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- Connection between Nash w/ TCs $\breve{\mu}_\Lambda$ and Nash w/o TCs $\breve{\mu}$:
  ✓ Equal risk tolerances (or $N = 2$) and as $\Lambda \downarrow \Longrightarrow \breve{\mu}_\Lambda$ reverts to $\breve{\mu}$.
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Closing remarks

Summary of this work

- Impose a rather simple price impact model when sharing risk is the main motivation for trade.
- We see the equilibrium returns as a function of the revealed hedging needs for each investor.
- Equilibrium is nice and clear without transaction costs.
- But thin markets do have transaction costs.
- We get an explicit solution of the best-response and the Nash equilibrium returns when investors are two or they have the same risk aversion.
- When noise traders are absent, equilibrium returns stay untouched by the price impact.
- Under the presence of noise traders, equilibrium return is heavily affected by the investors' price impact.

✓ Our manuscript is available at ssrn under the title: *"Continuous-time Equilibrium in Markets with Price Impact & Transaction Costs"*

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