Department of Economics

Athens University of Economics and Business

WORKING PAPER no. 15-2024

Distributionally Conservative Stochastic Dominance via Subsampling

Stelios Arvanitis

October 2024

Πατησίων 76, 104 34 Αθήνα. Τηλ.: 210 8203303-5. E-mail: econ@aueb.gr / www.aueb.gr
76, Patission Street, Athens 104 34 Greece. Tel.: (+30) 210 8203303-5

The Working Papers in this series circulate mainly for early presentation and discussion, as well as for the information of the Academic Community and all interested in our current research activity.

The authors assume full responsibility for the accuracy of their paper as well as for the opinions expressed therein.

Distributionally Conservative Stochastic Dominance via Subsampling

Stelios Arvanitis*

October 20, 2024

Abstract

This note defines distributionaly concervative versions of stochastic dominance relations based on subsampling. It presents a non-asymptotic analysis of the probability of the False Dominance (FD) error for the empirical version of the subsampling based empirical dominance procedure. The analysis is based on the generalization of the McDiarmid concentration inequality to η-mixing processes by Kontorovich and Ramanan (2008). The concentration bounds obtained depend on "entropy" characteristics of the problem, like the Lipschitz coefficients of the utilities involved, on the FD parameters involved, as well as on the coefficients that represent temporal dependence at each subsample. The analysis establishes tighter concentration bounds for the conservative procedure in both stationary and non-stationary cases when the subsampling rate is appropriately chosen.

JEL Codes: C44, C58, G11. *MSC2020*: 62C, 62P20, 62E17.

Keywords: Stochastic dominance, subsampling, distributional conservativeness, concentration inequality, non-asymptotic analysis, false dominance classification.

1 Introduction

Stochastic dominance relations serve as (pre-) orders on collections of probability distributions over the real line (refer to Fishburn (1976) [4]) for precise definitions and Shah (2017) [16] for extensions into function spaces). These relations are typically defined by inequalities involving sets of utility

*Department of Economics, Athens University of Economics and Business, Greece. Email: stelios@aueb.gr

functions within the expected utility framework (see Fishburn (1976) [4], as well as Levy (1992) [8]). Then, dominance concerning such a relation indicates preference determined by every utility within the class, and vice versa. Consequently, these relations are inherently robust regarding preferences; they represent maximally conservative decisions under the uncertainty concerning preferences about risk.

Their research has become significant in the fields of economics, finance, operations research, and statistics/econometrics (see, for example, McFadden (1989) [10], Mosler and Scarsini (1993) [11], Gayant and Le Pape (2017) [5]), as it, among others, allows for inference about properties of optimal choice without necessitating a parametric specification for preferences.

An illustrative case in finance is portfolio optimization based on stochastic dominance. This approach extends the traditional mean-variance method without depending on assumptions of satiation or elliptical return distributions. Recent studies in operations research and econometrics have tackled the analytical hurdles of numerical optimization and statistical inference, thereby enhancing its practical applicability. Notable applications are highlighted by Hodder et al. (2015) [6]; Constantinides et al. (2020) [3]; Post and Rodriguez-Longarela (2021) [13]; and Arvanitis and Post (2024) [2]. The process of selecting a portfolio generally involves optimizing an empirical criterion, constrained by the requirement that the chosen portfolios empirically dominate a benchmark portfolio. Although the portfolio deemed empirically optimal dominates the benchmark within the sample, it remains vulnerable to the potential decision error of False Dominance (FD) classification when applied to the entire population.

Within general sampling schemes, this decision error becomes asymptotically negligible. Controlling the probability of this error with a fixed sample size is crucial in practice, especially when the sample size is relatively small compared to the dimensionality of the portfolios analyzed. A potential way of achieving this is by introducing forms of distributional robustness in the problem; this can be done via uncertainty sets containing distributions close to the empirical one for conservative evaluations of the empirical dominance conditions.

The present note investigates a path where the uncertainty sets are constructed via subsampling. Specifically, given the sample, and the choice of a subsampling rate by the analyst, the empirical dominance conditions are evaluated at the empirical distributions of the maximally overlapping subsamples. Then dominance occurs if and only if those are favourable for every utility in the class as well as for every subsample considered.

A by-product of this construction is that such a distributionally conservative dominance relation can be also defined in the population by integrating the subsampling enhanced dominance relations w.r.t. the latent DGP distribution. When the DGP is stationary, this reduces the relation to a standard stochastic dominance relation involving the marginal stationary distribution of the process. In the context of non-stationarity however, it provides for example a way of constructing conservative population stochastic dominance relations for processes that exhibit structural changes. Population dominance then occurs if the usual dominance holds at every epoch of the process, or if the epochs at which it fails become asymptotically negligible with the sample size.

The note performs the non-asymptotic investigation of the probability of the FD error for the empirical version of the subsampling based dominance using the technology of concentration inequalities. Specifically, the analysis is based on the generalization of the McDiarmid concentration inequality (see McDiarmid (1989) [9]) to η-mixing processes by Kontorovich and Ramanan (2008) [7]. The concentration bounds obtained depend on "entropy" characteristics of the problem, like the Lipschitz coefficients of the utilities involved, on the FD parameters involved, as well as on the coefficients that represent temporal dependence at each subsample. The analysis establishes that those bounds can be tighter for the subsampling based conservative procedure, both in the non-stationary case due to the dependence of the bound on the subsample of minimal temporal dependence, or in the stationary scenario as long as the subsampling rate is approproriately chosen.

The structure of the remaining note is the following: the second section fixes the general notation and framework and provides the definitions and properties of the subsampling based distributionally conservative versions of stochastic dominance relations. The third section derives and analyzes the non-asymptotic bounds for the probability of FD. The final section concludes and briefly discusses paths for future research.

2 Conservative stochastic dominance based on subsampling

The random element $\mathbf{X}_t$, assumes its values in $\mathcal{X}$, a countable and bounded subset of $\mathbb{R}^N$; $\operatorname{diam}(\mathcal{X})$ denotes its Euclidean diameter. From an application perspective, $\mathcal{X}$ may carry some economic significance, such as representing potential returns for N base financial assets. In this context, the aspects of boundedness and countability are pertinent to such empirical data. The stochastic process $(\mathbf{X}_t)_{t\in\mathbb{N}}$, that arises, may be viewed as capturing the discrete-time stochastic dynamics of the aforementioned returns. It is presumed to adhere to a specific and latent distribution $\mathbb{P}$ with respect to which stationarity is not a prerequisite. Given a sample $(\mathbf{X}_t)_{1\leq t\leq T}$, available to the analyst and the subsampling rate $1 < b_T \leq T$, the maximally overlapping subsamples $(\mathbf{X}\,(m)_t)_{1\leq m\leq T-b_T+1, m\leq t\leq b_T+m-1}$ are considered; there $\mathbb{F}_{m,b_T}$ denotes the empirical distribution of $(\mathbf{X}\,(m)_t)_{m\leq t\leq b_T+m-1}$.

Λ is a closed subset of the $N-1$ standard simplex. An arbitrary element of Λ is denoted by $\boldsymbol{\lambda}$ and, in the financial context mentioned earlier, can be interpreted as a convex portfolio created from the base assets, leading to returns of $\boldsymbol{\lambda}'\mathbf{X}_t$. Within Λ, $\boldsymbol{\tau}$ is a distinguished element that could represent a benchmark portfolio relevant to the analyst's research questions.

$\mathcal{U}$ is a family of Lipschitz continuous real utility functions that represent preference relations over the set of probability distributions defined on $\mathcal{X}$. The scaling invariance of the preferences represented by the elements of $\mathcal{U}$, implies that the family can be without loss of generality considered as uniformly Lipschitz with a common coefficient denoted by $l_{\mathcal{U}} > 0$. A prominent example associated with the concept of second order stochastic dominance is the set of Russell-Seo utilities, $\{u(x) := -(z-x)_+,\ z\in\mathcal{X}\}$, see Russell and Seo (1989) [14]; there $l_{\mathcal{U}} = 1$. Given a probability measure $\mathbb{F}$ on $\mathcal{X}$ and a $u\in\mathcal{U}$, $G_{\mathbb{F}}(u,\boldsymbol{\lambda},\boldsymbol{\tau}) := \mathbb{E}_{\mathbb{F}}\left[u\left(\boldsymbol{\lambda}'\mathbf{X}\right)\right] - \mathbb{E}_{\mathbb{F}}\left[u\left(\boldsymbol{\tau}'\mathbf{X}\right)\right]$ is then the expected utility differential between the projects $\boldsymbol{\lambda}$ and $\boldsymbol{\tau}$, evaluated at $(\mathbb{F}, u)$.

The tuple $(\mathcal{U}, \mathbb{P}, b_T, (\mathbf{X}_t)_{t\in\mathbb{N}})$ defines then the following stochastic dominance relation:

Definition 1. *$\boldsymbol{\lambda}$ dominates the benchmark $\boldsymbol{\tau}$ w.r.t. $(\mathcal{U}, \mathbb{P}, b_T, (\mathbf{X}_t)_{t\in\mathbb{N}})$ iff $\mathbb{E}_{\mathbb{P}}\left[G_{\mathbb{F}_{m,b_T}}(u,\boldsymbol{\lambda},\boldsymbol{\tau})\right] \geq 0$, $\forall u\in\mathcal{U}$, $\forall m = 1,\ldots,T-b_T-1$.*

The definition asserts that dominance occurs if and only if $\boldsymbol{\lambda}$ is preferred to the benchmark in terms of the expected w.r.t. $\mathbb{P}$, average utility of the m^{th} subsample, by every utility in the class

$\mathcal{U}$, and every subsample defined by the choice of the rate b_T. The resulting relation also adheres to Definition 5.1.(1) of Shah (2017) [16]. Within the framework outlined in this paper, the state space is the convex hull of the set $\left\{\boldsymbol{\lambda}'\mathbf{x}, \boldsymbol{\lambda} \in \ , \mathbf{x} \in \mathcal{X}\right\}^T$, while the function space of utilities is defined as the set $\left\{\frac{1}{b_T}\sum_{t=m}^{b_T+m-1} u(x_t),\ u \in \mathcal{U}, m = 1, \ldots, b_T + m - 1\right\}$. Thus, the relation can also be perceived as an ordering between stochastic processes supported on the countable set $\mathcal{X}^T$.

When the process $(\mathbf{X}_t)_{t\in\mathbb{N}}$ is stationary, the subsampling part of the relation becomes irrelevant, since then $\mathbb{E}_{\mathbb{P}}\left[G_{\mathbb{F}_{m,b_T}}(u, \boldsymbol{\lambda}, \boldsymbol{\tau})\right] = \mathbb{E}_{\mathbb{F}}\left[u\left(\boldsymbol{\lambda}'\mathbf{X}\right)\right] - \mathbb{E}_{\mathbb{F}}\left[u\left(\boldsymbol{\tau}'\mathbf{X}\right)\right]$, where $\mathbb{F}$ now denotes the stationary marginals of $\mathbb{P}$. The definition thus reduces to standard forms of stochastic dominance relations over sets of probability distributions on $\mathbb{R}$; when for example $\mathcal{U}$ is the set of Russell-Seo utilities, the usual second order stochastic dominance relation is recovered (see Russell and Seo (1989) [14]).

The subsampling rate gains importance in the context of non-stationarity, for example due to heterogeneity among the marginal distributions within $\mathbb{P}$. Then averaging of expected utility differentials inside the subsamples becomes important. For a simple example, suppose that the process $(\mathbf{X}_t)_{t\in\mathbb{N}}$ is comprised by two different epochs: for some $1 < T^\star < T$, $\mathbf{X}_t = \begin{cases} \mathbf{X}_t^{(1)},\ t \leq T^\star \\ \mathbf{X}_t^{(2)},\ t \leq T^\star \end{cases}$, where $(\mathbf{X}_t^{(1)})_{t\in\mathbb{N}}$ and $(\mathbf{X}_t^{(2)})_{t\in\mathbb{N}}$ are two mutually independent stationary $\mathcal{X}$-valued processes, with stationary marginals $\mathbb{F}_1$ and $\mathbb{F}_2$ respectively. Suppose also that $\mathbb{E}_{\mathbb{F}_2}\left[u\left(\boldsymbol{\lambda}'\mathbf{X}\right)\right] - \mathbb{E}_{\mathbb{F}_2}\left[u\left(\boldsymbol{\tau}'\mathbf{X}\right)\right] \geq 0$ for every utility in the class, but there exists some $v \in \mathcal{U}$ for which $\mathbb{E}_{\mathbb{F}_1}\left[v\left(\boldsymbol{\lambda}'\mathbf{X}\right)\right] - \mathbb{E}_{\mathbb{F}_1}\left[v\left(\boldsymbol{\tau}'\mathbf{X}\right)\right] < 0$, i.e. in the initial epoch there exists a utility that strictly prefers the benchmark, something not true after the structural change. When $T^\star|G_{\mathbb{F}_1}(v, \boldsymbol{\lambda}, \boldsymbol{\tau})| > (b_T - T^\star)G_{\mathbb{F}_2}(v, \boldsymbol{\lambda}, \boldsymbol{\tau})$, then $\boldsymbol{\lambda}$ cannot be considered dominant to the benchmark by the dominance relation, since there will always exist a subsample-namely the first-for which the definition would fail; for the particular choice of b_T, the first epoch expected utility relations are of importance to the analyst as long as the previous inequality is satisfied. If as $T \to \infty$, $b_T \to \infty$, and $G_{\mathbb{F}_2}(v, \boldsymbol{\lambda}, \boldsymbol{\tau}) > 0$, then eventually $b_T > \frac{T^\star(|G_{\mathbb{F}_1}(v,\boldsymbol{\lambda},\boldsymbol{\tau})| + G_{\mathbb{F}_2}(v,\boldsymbol{\lambda},\boldsymbol{\tau}))}{G_{\mathbb{F}_2}(v,\boldsymbol{\lambda},\boldsymbol{\tau})}$, and the first epoch expected utility relations become negligible for the researcher; dominance eventually holds. Hence, the subsampling part of the definition allows for scrutiny w.r.t. the potentially different expected utility relations at different epochs in the sample, somehow justifying the characterization distributionally robust. The choice of the subsampling rate reflects among others the analysts' preferences on the level of epoch refinement.

The latency of $\mathbb{P}$ generally implies the latency of the relation for the analyst. An empirical approximation of the relation is the one produced when the outer integration w.r.t. $\mathbb{P}$ is dropped from the previous definition:

Definition 2. $\boldsymbol{\lambda}$ *empirically dominates the benchmark* $\boldsymbol{\tau}$ *w.r.t.* $(\mathcal{U}, b_T, (\mathbf{X}_t)_{t\in\mathbb{N}})$ *iff* $G_{\mathbb{F}_{m,b_T}}(u, \boldsymbol{\lambda}, \boldsymbol{\tau}) \geq 0$*,* $\forall u \in \mathcal{U}$*,* $\forall m = 1, \ldots, T - b_T - 1$*.*

What matters now for the empirical dominance are the relations between the average utilities for every subsample at the sample realization. Given that $G_{\mathbb{F}_{m,b_T}}$ generally depends on m and b_T, the subsampling part of the definition plays a role even in cases where the underlying process is stationary, unless $b_T = T$. As a matter of fact, in the context of stationarity, and if the subsampling distributions $\mathbb{F}_{m,b_T}$ are stochastic approximations of the unknown marginal $\mathbb{F}$, the empirical version of the definition means that now dominance is defined in a conservative and distributionally robust manner when $b_T < T$: the expected utility relations must hold for every allowed subsample empirical distribution.

The non-empty subset of Λ that is comprised by elements that empirically dominate the benchmark $\boldsymbol{\tau}$ is obtainable as the set of solutions of the following statistical-variational problem:

$$\inf_{m} \inf_{u\in\mathcal{U}} G_{\mathbb{F}_{m,b_T}}(u, \boldsymbol{\lambda}, \boldsymbol{\tau}) \geq 0, \tag{1}$$

for $m = 1, \ldots, T - b_T - 1$.

Any non-trivial solution of (1) can be subsequently utilized in further optimization procedures. For example, in the financial context of portfolio choice, the optimal $\boldsymbol{\lambda}$ is usually chosen as the optimizer of some further empirical criterion defined on the set of solutions of stochastic dominance problems like (1)-see for example Arvanitis and Post (2024) [2]. Therefore, it is essential that the probability of the decision error event of false dominance (FD)—where an empirical dominant $\boldsymbol{\lambda}$ is incorrectly identified as population dominant—is minimal.

3 Concentration and the false dominance probability bound

In this section a non-asymptotic analysis of the probability of FD for an arbitrary non-trivial solution of (1) is provided. Specifically, $\boldsymbol{\lambda}(b_T)$, satisfies $\inf_m \inf_{u\in\mathcal{U}} G_{\mathbb{F}_{m,b_T}}(u,\boldsymbol{\lambda}(b_T),\boldsymbol{\tau}) \geq 0$, while being non-dominant in the population; there exists a "false dominance" parameter $\Psi < 0$, a $u \in \mathcal{U}$, and a subsample m, such that $\mathbb{E}_{\mathbb{P}}\left[G_{\mathbb{F}_{m,b_T}}(u,\boldsymbol{\lambda}(b_T),\boldsymbol{\tau})\right] < \Psi$, where $\mathbb{E}_{\mathbb{P}}\left[G_{\mathbb{F}_{m,b_T}}(u,\boldsymbol{\lambda}(b_T),\boldsymbol{\tau})\right] := \mathbb{E}_{\mathbb{P}}\left[G_{\mathbb{F}_{m,b_T}}(u,\boldsymbol{\lambda},\boldsymbol{\tau})\right]|_{\boldsymbol{\lambda}=\boldsymbol{\lambda}(b_T)}$. The probability of this event is expected to be influenced by various factors, including the subsampling rate and the set of related subsamples. Analyzing these can help determine if the conservative method outlined in (1) is linked to a lower probability of FD for certain subsampling parameter selections, as opposed to the non-conservative full sample scenario where $b_T = T$.

The analysis of the probability of FD is based on a generalization of the McDiarmid concentration inequality (see McDiarmid (1989) [9]) from the framework of independence to the one of η-mixing by Kontorovich and Ramanan (2008) [7]. For the precise definition of the concept the interested reader is referred to the previous paper. The formal definition involves bounds in the form of the η-mixing coefficients; for the m^{th} subsample , the coefficient $\bar{\eta}_{t,s}$, $m \leq t \leq s \leq b_T+m-1$ is the least upper bound of the maximal total variation distance between the conditional distributions of $(\mathbf{X}_j)_{j=s,\ldots,b_T+m-1}$ given $(\mathbf{X}_j)_{j=m,\ldots,t-1}$ when $t < s$, and equals 1 when $t = s$. Those coefficients are summarized in the $b_T \times b_T$ triangular matrix Δ_{m,b_T}, with $\Delta_{m,b_T}(t,s) := \begin{cases} \bar{\eta}_{t,s},\ t \leq s \\ 0,\ t > s \end{cases}$; $\|\Delta_{m,b_T}\|_\infty$ denotes its operator norm which equals $1 + \max_{m\leq t<b_T+m-1}\sum_{j=t+1}^{b_T+m-1}\bar{\eta}_{t,j}$. The process $(\mathbf{X}_t)_{t\in\mathbb{N}}$ is then characterized as η-mixing iff $\sup_T \|\Delta_{1,T}\|_\infty < +\infty$. Examples of η-mixing processes are contractive Markov processes as well as geometrically ϕ (uniform)-mixing processes-see for example Kontorovich and Ramanan (2008) [7] and Samson (2000) [15].

The FD analysis proceeds by considering the supposedly $\mathbb{P}$-positive probability event $\mathcal{E}_\Psi^{(b_T)} := \left\{\mathbb{E}_{\mathbb{P}}\left[G_{\mathbb{F}_{m,b_T}}(u,\boldsymbol{\lambda}(b_T),\boldsymbol{\tau})\right] < \Psi < 0,\ \exists(u,m)\right\}$ indicating population nondominance with parameter Ψ. Given this, the probability of FD is formed as $\mathbb{P}\left[\inf_m \inf_{u\in\mathcal{U}} G_{\mathbb{F}_{m,b_T}}(u,\boldsymbol{\lambda}(b_T),\boldsymbol{\tau}) \geq 0/\mathcal{E}_\Psi^{(b_T)}\right]$; then the following result is obtained:

Proposition 3. *Suppose that the stochastic process* $(\mathbf{X}_t)_{t\in\mathbb{N}}$ *is* η*-mixing w.r.t.* $\mathbb{P}$*. Let* $K_{\mathbb{P},m} := \inf_m \mathbb{E}_{\mathbb{P}}\left[G_{\mathbb{F}_{m,b_T}}(u,\boldsymbol{\lambda}(b_T),\boldsymbol{\tau})\right] - \mathbb{E}_{\mathbb{P}}\left[G_{\mathbb{F}_{m,b_T}}(u,\boldsymbol{\lambda}(b_T),\boldsymbol{\tau})\right]$*. Then, for any* $T \geq 1$ *and* $1 \leq b_T \leq T$*, it holds that*

$$\mathbb{P}\left[\inf_m \inf_{u\in\mathcal{U}} G_{\mathbb{F}_{m,b_T}}(u,\boldsymbol{\lambda}(b_T),\boldsymbol{\tau}) \geq 0/\mathcal{E}_{\Psi}^{(b_T)}\right] \leq \exp\left(-\frac{b_T \max\left\{\Psi^2,(|\Psi|+\inf_m K_{\mathbb{P},m})^2\right\}}{16 l_{\mathcal{U}}^2 \|\Lambda\|_\infty^2 \mathrm{diam}^2\left(\mathcal{X}\right) \inf_m \|\Delta_{m,b_T}\|_\infty^2}\right). \quad (2)$$

Furthermore, for any $\alpha \in (0,1)$*, whenever*

$$\max\left\{|\Psi|,||\Psi| + \inf_m K_{\mathbb{P},m}|\right\} > 4 l_{\mathcal{U}}\|\Lambda\|_\infty \mathrm{diam}\left(\mathcal{X}\right) \inf_m \|\Delta_{m,b_T}\|_\infty \sqrt{\ln \alpha^{b_T^{-1}}},$$

it holds that

$$\mathbb{P}\left[\inf_m \inf_{u\in\mathcal{U}} G_{\mathbb{F}_{m,b_T}}(u,\boldsymbol{\lambda}(b_T),\boldsymbol{\tau}) \geq 0/\mathcal{E}_{\Psi}^{(b_T)}\right] \leq \alpha. \quad (3)$$

Proof. In what follows d_H denotes the usual Hamming metric restricted on $\mathcal{X}$; this is defined as $d_H(\boldsymbol{x},\boldsymbol{y}) := \sum_{i=1}^N \mathbf{1}_{\boldsymbol{x}_i \neq \boldsymbol{y}_i}$, for arbitrary $\boldsymbol{x},\boldsymbol{y} \in \mathcal{X}$. For $\mathbf{x}_{m,b_T}$, $\mathbf{y}_{m,b_T}$, realizations of $\left(\mathbf{X}\left(m\right)_t\right)_{m\leq t\leq b_T}$ and due to Holder's inequality it is obtained that

$$\begin{aligned}\left|\mathbb{E}_{\mathbb{F}_{m,b_T}}\left[u\left(\boldsymbol{\lambda}'\mathbf{x}_{m,b_T}\right)\right] - \mathbb{E}_{\mathbb{F}_{m,b_T}}\left[u\left(\boldsymbol{\lambda}'\mathbf{y}_{m,b_T}\right)\right]\right| &\leq \tfrac{l_{\mathcal{U}}\|\Lambda\|_\infty}{b_T}\left\|\mathbf{x}_{m,b_T} - \mathbf{y}_{m,b_T}\right\|_1 \\ &\leq d_H\left(\mathbf{x}_{m,b_T},\mathbf{y}_{m,b_T}\right)\tfrac{l_{\mathcal{U}}\|\Lambda\|_\infty}{b_T}\mathrm{diam}\left(\mathcal{X}\right).\end{aligned}$$

This and the triangle inequality imply that

$$\left|G_{\mathbb{F}_{m,b_T}}^{\mathbf{x}_{m,b_T}}(u,\boldsymbol{\lambda},\boldsymbol{\tau}) - G_{\mathbb{F}_{m,b_T}}^{\mathbf{y}_{m,b_T}}(u,\boldsymbol{\lambda},\boldsymbol{\tau})\right| \leq d_H\left(\mathbf{x}_{m,b_T},\mathbf{y}_{m,b_T}\right)\tfrac{2l_{\mathcal{U}}\|\Lambda\|_\infty}{b_T}\mathrm{diam}\left(\mathcal{X}\right),$$

where $G^{\mathbf{x}_{m,b_T}}$ denotes evaluation of the expected utility differential at realization $\mathbf{x}_{m,b_T}$.

Now, from the definitions of $\boldsymbol{\lambda}(b_T)$ and the event $\mathcal{E}_{\Psi}^{(b_T)}$ it is obtained that

$$
\begin{aligned}
&\mathbb{P}\left[\inf_m \inf_{u\in\mathcal{U}} G_{\mathbb{F}_{m,b_T}}(u,\boldsymbol{\lambda}(b_T),\boldsymbol{\tau})) \geq 0/\mathcal{E}_{\Psi}^{(b_T)}\right] \\
&\leq \mathbb{P}\left[\inf_m \left[G_{\mathbb{F}_{m,b_T}}(u,\boldsymbol{\lambda}(b_T),\boldsymbol{\tau})\right] \geq -\Psi + \inf_m \mathbb{E}_{\mathbb{P}}\left[G_{\mathbb{F}_{m,b_T}}(u,\boldsymbol{\lambda}(b_T),\boldsymbol{\tau})/\mathcal{E}_{\Psi}^{(b_T)}\right]\right] \\
&\overset{(1)}{\leq} \inf_m \mathbb{P}\left[G_{\mathbb{F}_{m,b_T}}(u,\boldsymbol{\lambda}(b_T),\boldsymbol{\tau}) \geq -\Psi + \inf_m \mathbb{E}_{\mathbb{P}}\left[G_{\mathbb{F}_{m,b_T}}(u,\boldsymbol{\lambda}(b_T),\boldsymbol{\tau})/\mathcal{E}_{\Psi}^{(b_T)}\right]\right] \\
&\leq \inf_m \mathbb{P}\left[G_{\mathbb{F}_{m,b_T}}(u,\boldsymbol{\lambda}(b_T),\boldsymbol{\tau}) - \mathbb{E}_{\mathbb{P}}\left[G_{\mathbb{F}_{m,b_T}}(u,\boldsymbol{\lambda}(b_T),\boldsymbol{\tau})\right] \geq -\Psi + K_{\mathbb{P},m}/\mathcal{E}_{\Psi}^{(b_T)}\right] \\
&\overset{(2)}{\leq} \inf_m \exp\left(-\frac{b_T(\Psi+K_{\mathbb{P},m})^2}{16 l_{\mathcal{U}}^2\|\Lambda\|_\infty^2 \mathrm{diam}^2(\mathcal{X})\|\Delta_{m,b_T}\|_\infty^2}\right) \overset{(3)}{=} \exp\left(-\frac{b_T \max\left\{|\Psi|^2,(|\Psi|+\inf_m K_{\mathbb{P},m})^2\right\}}{16 l_{\mathcal{U}}^2\|\Lambda\|_\infty^2 \mathrm{diam}^2(\mathcal{X}) \inf_m \|\Delta_{m,b_T}\|_\infty^2}\right),
\end{aligned}
$$

where $\overset{(1)}{\leq}$ follows from the Fréchet inequality, the concentration in $\overset{(2)}{\leq}$ follows from Theorem 1.1 of Kontorovich and Ramanan (2008) [7], for $c = \frac{2l_{\mathcal{U}}\|\Lambda\|_\infty}{b_T}\mathrm{diam}(\mathcal{X})$ and $n = b_T$, and $\overset{(3)}{=}$ follows by optimization, and Lemma 7.64 of Aliprantis and Border (1999) [1]. (3) follows by bounding the exponential concentration above by α and solving for $\max\{|\Psi|, (|\Psi| + \inf_m K_{\mathbb{P},m})\}$. □

The tightness of the exponential concentration bound that appears in (3) depends positively on the subsampling rate b_T. It also depends positively on the false dominance parameter $|\Psi|$, and the magnitude of the difference $\sup_m \mathbb{E}_{\mathbb{P}}\left[G_{\mathbb{F}_{m,b_T}}(u,\boldsymbol{\lambda}(b_T)\right] - \inf_m \mathbb{E}_{\mathbb{P}}\left[G_{\mathbb{F}_{m,b_T}}(u,\boldsymbol{\lambda}(b_T)\right]$, whenever this is greater than $|\Psi|$ in which case the term $(|\Psi|+\inf_m K_{\mathbb{P},m})^2$ dominates. This provides a motivation for applying the subsampling procedure whenever the process $(\mathbf{X}_t)_{t\in\mathbb{N}}$ is non stationary; choosing $b_T < T$ could ceteris paribus imply small probability of FD, if the aforementioned difference is large enough even for moderate values of the FD parameter. Notice that whenever $b_T = T$, or the process is stationary, this difference is nullified.

Furthermore, and as expected, the tightness of the bound depends negatively on the "entropy" parameters $l_{\mathcal{U}}$, $\|\Lambda\|_\infty$, and $\mathrm{diam}(\mathcal{X})$ that represent the complexity of the function class involved in the statistical procedure in (1). Leveraging the scaling invariance of $\mathcal{U}$ to make $l_{\mathcal{U}}$ arbitrarily small is futile; this would simultaneously require the same rescalling of the FD parameter Ψ as well as the term $\inf_m K_{\mathbb{P},m}$. Since Λ is part of the standard simplex, $\|\Lambda\|_\infty$ can be further bounded by 1.

The bound tightness also depends on the minimal w.r.t. the set of subsamples norm of the mixing coefficients of the process. Given the choice of the subsampling rate, and in the conservative case where $b_T < T$, dependence of the bound on the optimal temporal mixing structure across the subsamples provides another motivation for the consideration of the conservative subsampling procedure compared to the full sample one; given that the full sample bound equals $\exp\left(-\frac{T\Psi^2}{16 l_{16\mathcal{U}}^2 \|\Lambda\|_\infty^2 \mathrm{diam}^2(\mathcal{X})\|\Delta_{1,T}\|_\infty^2}\right)$, and even if $|\Psi|$ dominates the numerator term, the subsampling bound is tighter if $b_T > T\frac{\inf_m \|\Delta_{m,b_T}\|_\infty}{\|\Delta_{1,T}\|_\infty}$. Forms of non-stationarity can provide leeway for the optimal choice of the subsampling rate as it relates this choice to the minimal-across the resulting subsamples-temporal dependence; this can also provide more room in order for the numerator term to be optimal due to the dominance of the $(|\Psi| + \inf_m K_{\mathbb{P},m})^2$ term.

For a simple example, consider the two-epoch process of the previous section. Furthermore, suppose that $(\mathbf{X}_t^{(1)})_{t\in\mathbb{N}}$ and $(\mathbf{X}_t^{(2)})_{t\in\mathbb{N}}$ are η-mixing; $\bar{\eta}_1$ and $\bar{\eta}_2$ denote the respective mixing coefficients. It is then establishable that $(\mathbf{X}_t)_{t\in\mathbb{N}}$ is also η-mixing, with mixing coefficients $\bar{\eta}_{t,s} = \begin{cases} \bar{\eta}_{1;t,s},\ b_T + m - 1 < T^\star \\ 0,\ t < T^\star \leq s \\ \bar{\eta}_{2;t,s},\ T^\star \leq t \end{cases}$. It then holds that $\|\Delta_{1,T}\|_\infty := 1 + \max\left\{\sum_{j=2}^{T^\star-1}\bar{\eta}_{1;1,j}, \sum_{j=T^\star+1}^{T}\bar{\eta}_{2;T^\star,j}\right\}$, and $\inf_m \|\Delta_{m,b_T}\|_\infty = \begin{cases} 1 + \min\left\{\bar{\eta}_{1;T^\star-2,T^\star-1}, \min_{m\geq T^\star}\sum_{j=m+1}^{b_T+m-1}\bar{\eta}_{2;m,j}\right\},\ b_T < T^\star \\ 1 + \bar{\eta}_{1;T^\star-2,T^\star-1},\ b_T \geq T^\star \end{cases}$. Then the latter is smaller than the former.

Stationarity reduces the $b_T > T\frac{\inf_m \|\Delta_{m,b_T}\|_\infty}{\|\Delta_{1,T}\|_\infty}$ inequality to the more restrictive version: $\frac{b_T}{T} > \frac{1+\max_{1\leq t\leq b_T-1}\sum_{j=t+1}^{b_T}\bar{\eta}_{t,j}}{1+\max_{1\leq t\leq T-1}\sum_{j=t+1}^{T}\bar{\eta}_{t,j}}$. If furthermore for some $0 \leq \eta < 1$, whenever $t \leq s$, $\bar{\eta}_{t,s} \leq c\eta^{s-t}$, as in the case of geometric ϕ-mixing or a uniformly contracting Markov process-see for example Samson (2000) [15], revisiting the concentration bounds leads to the sufficient condition for tighter subsampling bound $\frac{b_T}{T} > \frac{1-\eta^{b_T}}{1-\eta^T}$. If, in addition, b_T assumes the form T^δ for some $0 < \delta < 1$, then the choice of any δ that also satisfies $T^{\delta-1} > \frac{1-\eta^{T^\delta}}{1-\eta^T}$, implies a tighter bound for the subsampling procedure. For large enough T, this holds whenever $\ln(1-\delta) < \eta^{T^\delta} - \eta^T$. Hence, the conservative subsampling empirical dominance procedure can be also useful in cases of stationarity; conservativeness implies that the optimal choice of the subsampling rate is connected to tighter bounds for the probability of FD.

The analysis leading to (3) provides also a lower bound for the subsampling rate in order for the procedure to have a finite sample probability of FD less than a given significance level. Combining the result with the previous discussion it is obtained that if it is true that $\max\left\{T\frac{\inf_m \|\Delta_{m,b_T}\|_\infty}{\|\Delta_{1,T}\|_\infty}, -\frac{16\ln\alpha l_{\mathcal{U}}^2\|\Lambda\|_\infty^2 \mathrm{diam}^2(\mathcal{X})\inf_m \|\Delta_{m,b_T}\|_\infty^2}{\max\left\{|\Psi|^2,(|\Psi|+\inf_m K_{\mathbb{P},m})^2\right\}}\right\} < b_T < T$, then the conservative subsampling procedure simultaneously achieves the nominal significance, while obtaining a lower FD probability bound compared to the full sample procedure.

Finally, if $\frac{16 l_{\mathcal{U}}^2\|\Lambda\|_\infty^2 \mathrm{diam}^2(\mathcal{X})\inf_m \|\Delta_{m,b_T}\|_\infty^2}{\max\left\{|\Psi|^2,(|\Psi|+\inf_m K_{\mathbb{P},m})^2\right\}} = o(b_T)$, and if as $T\to\infty$, $b_T\to\infty$, then it follows that $\lim_{T\to\infty}\mathbb{P}\left[\inf_m \inf_{u\in\mathcal{U}} G_{\mathbb{F}_{m,b_T}}(u,\boldsymbol{\lambda}(b_T),\boldsymbol{\tau}) \geq 0/\mathcal{E}_\Psi^{(b_T)}\right] = 0$, indicating an asymptotically negligible probability for the FD decision error. This also allows for the dependence on T of the parameters Ψ and N.

4 Discussion

The results above motivate both the considerations of the conservative subsampling based dominance definition, as well as the subsampling based statistical procedure for the estimation of dominant elements. In nonstationary frameworks involving, for example, structural breaks between η-mixing processes the subsampling-based definition of the dominance relation can be related to the detection of dominance relations that persist across different regimes for the process involved. Under η-mixing the subsampling based estimation can be related to smaller concentration bounds for the probability of the decision error of false dominance when the subsampling rate can be chosen so as to take into account the minimal across the resulting subsamples temporal dependence structure. In stationary frameworks the subsampling based dominance definition coincides with the classical one. Even then, the subsampling-based estimation can be related to smaller concentration bounds whenever the choice of the susampling rate takes into account the way the mixing coefficients evolve with the sample size.

The results on the probability bounds are based on strong assumptions about the support of the random elements involved in the constructions, as well as the form of the temporal dependence in the underlying stochastic process. Extensions of the results in cases where the supports are not necessarily countable and/or the temporal dependence assumes the more familiar form of strong mixing are paths of further research. Such considerations could be facilitated by fascinating results relating Ricci (or more generally Bakry-Émery) positive curvature-dimension conditions with transportation inequalities and concentration phenomena-see for example Villani (2009) [17] and Ohta and Takatsu (2011) [12].

References

[1] Aliprantis, C. D., & Border, K. C., 1999. Infinite Dimensional Analysis, A Hitchhiker's Guide (2nd ed.). Springer.

[2] Arvanitis, S. and Post, T., 2024. Generalized stochastic arbitrage opportunities. Management Science, 70(7), pp.4629-4648.

[3] Constantinides, G.M.; Czerwonko, M.; Perrakis, S. Mispriced Index Option Portfolios. Financ. Manag. 2020, 49, 297–330.

[4] Fishburn, P. C., 1976. Continua of stochastic dominance relations for bounded probability distributions, Journal of Mathematical Economics, 3(3), 295-311.

[5] Gayant, J. P., & Le Pape, N., 2017. Increasing degree inequality, Journal of Mathematical Economics, 70, 185-189

[6] Hodder, J.E.; Jackwerth, J.C.; Kolokolova, O. Improved Portfolio Choice Using Second-Order Stochastic Dominance. Rev. Financ. 2015, 19, 1623–1647.

[7] Kontorovich, L. and Ramanan, K., 2008. Concentration inequalities for dependent random variables via the martingale method.

[8] Levy, H., 1992. Stochastic Dominance and Expected Utility: Survey and Analysis, Management Science, 38, 555–593.

[9] McDiarmid, C., 1989. On the method of bounded differences. Surveys in combinatorics, 141(1), pp.148-188.

[10] McFadden, D., 1989. Testing for Stochastic Dominance, in Studies in the Economics of Uncertainty, eds. T. Fomby and T. Seo, New York: Springer-Verlag, pp. 113–134.

[11] Mosler, K., & Scarsini, M., 1993. Stochastic Orders and Applications, a Classified Bibliography, Berlin: Springer-Verlag.

[12] Ohta, S.I. and Takatsu, A., 2011. Displacement convexity of generalized relative entropies. Advances in Mathematics, 228(3), pp.1742-1787.

[13] Post, T. and Longarela, I.R., 2021. Risk arbitrage opportunities for stock index options. Operations Research, 69(1), pp.100-113.

[14] Russell, W.R. and T.K. Seo, 1989, Representative sets for stochastic dominance rules, in: Studies in the economics of uncertainty: In honor of Josef Hadar, Springer-Verlag.

[15] Samson, P.M., 2000. Concentration of measure inequalities for Markov chains and Φ-mixing processes. The Annals of Probability, 28(1), pp.416-461.

[16] Shah, S. A. , 2017. How risky is a random process?, Journal of Mathematical Economics, 72, 70-81.

[17] Villani, C., 2009. Optimal transport: old and new (Vol. 338, p. 23). Berlin: springer.

Department of Economics
Athens University of Economics and Business

List of Recent Working Papers

2022

01-22 Is Ireland the most intangible intensive economy in Europe? A growth accounting perspective, Ilias Kostarakos, KieranMcQuinn and Petros Varthalitis
02-22 Common bank supervision and profitability convergence in the EU, Ioanna Avgeri, Yiannis Dendramis and Helen Louri
03-22 Missing Values in Panel Data Unit Root Tests, Yiannis Karavias, Elias Tzavalis and Haotian Zhang
04-22 Ordering Arbitrage Portfolios and Finding Arbitrage Opportunities, Stelios Arvanitis and Thierry Post
05-22 Concentration Inequalities for Kernel Density Estimators under Uniform Mixing, Stelios Arvanitis
06-22 Public Sector Corruption and the Valuation of Systemically Important Banks, Georgios Bertsatos, Spyros Pagratis, Plutarchos Sakellaris
07-22 Finance or Demand: What drives the Responses of Young and Small Firms to Financial Crises? Stelios Giannoulakis and Plutarchos Sakellaris
08-22 Production function estimation controlling for endogenous productivity disruptions, Plutarchos Sakellaris and Dimitris Zaverdas
09-22 A panel bounds testing procedure, Georgios Bertsatos, Plutarchos Sakellaris, Mike G. Tsionas
10-22 Social policy gone bad educationally: Unintended peer effects from transferred students, Christos Genakos and Eleni Kyrkopoulou
11-22 Inconsistency for the Gaussian QMLE in GARCH-type models with infinite variance, Stelios Arvanitis and Alexandros Louka
12-22 Time to question the wisdom of active monetary policies, George C. Bitros
13-22 Investors' Behavior in Cryptocurrency Market, Stelios Arvanitis, Nikolas Topaloglou and Georgios Tsomidis
14-22 On the asking price for selling Chelsea FC, Georgios Bertsatos and Gerassimos Sapountzoglou
15-22 Hysteresis, Financial Frictions and Monetary Policy, Konstantinos Giakas
16-22 Delay in Childbearing and the Evolution of Fertility Rates, Evangelos Dioikitopoulos and Dimitrios Varvarigos
17-22 Human capital threshold effects in economic development: A panel data approach with endogenous threshold, Dimitris Christopoulos, Dimitris Smyrnakis and Elias Tzavalis
18-22 Distributional aspects of rent seeking activities in a Real Business Cycle model, Tryfonas Christou, Apostolis Philippopoulos and Vanghelis Vassilatos

Πατησίων 76, 104 34 Αθήνα. Τηλ.: 210 8203303-5. E-mail: econ@aueb.gr / www.aueb.gr
76, Patission Street, Athens 104 34 Greece. Tel.: (+30) 210 8203303-5

2023

01-23 Real interest rate and monetary policy in the post Bretton Woods United States, George C. Bitros and Mara Vidali

02-23 Debt targets and fiscal consolidation in a two-country HANK model: the case of Euro Area, Xiaoshan Chen, Spyridon Lazarakis and Petros Varthalitis

03-23 Central bank digital currencies: Foundational issues and prospects looking forward, George C. Bitros and Anastasios G. Malliaris

04-23 The State and the Economy of Modern Greece. Key Drivers from 1821 to the Present, George Alogoskoufis

05-23 Sparse spanning portfolios and under-diversification with second-order stochastic dominance, Stelios Arvanitis, Olivier Scaillet, Nikolas Topaloglou

06-23 What makes for survival? Key characteristics of Greek incubated early-stage startup(per)s during the Crisis: a multivariate and machine learning approach, Ioannis Besis, Ioanna Sapfo Pepelasis and Spiros Paraskevas

07-23 The Twin Deficits, Monetary Instability and Debt Crises in the History of Modern Greece, George Alogoskoufis

08-23 Dealing with endogenous regressors using copulas; on the problem of near multicollinearity, Dimitris Christopoulos, Dimitris Smyrnakis and Elias Tzavalis

09-23 A machine learning approach to construct quarterly data on intangible investment for Eurozone, Angelos Alexopoulos and Petros Varthalitis

10-23 Asymmetries in Post-War Monetary Arrangements in Europe: From Bretton Woods to the Euro Area, George Alogoskoufis, Konstantinos Gravas and Laurent Jacque

11-23 Unanticipated Inflation, Unemployment Persistence and the New Keynesian Phillips Curve, George Alogoskoufis and Stelios Giannoulakis

12-23 Threshold Endogeneity in Threshold VARs: An Application to Monetary State Dependence, Dimitris Christopoulos, Peter McAdam and Elias Tzavalis

13-23 A DSGE Model for the European Unemployment Persistence, Konstantinos Giakas

14-23 Binary public decisions with a status quo: undominated mechanisms without coercion, Efthymios Athanasiou and Giacomo Valletta

15-23 Does Agents' learning explain deviations in the Euro Area between the Core and the Periphery? George Economides, Konstantinos Mavrigiannakis and Vanghelis Vassilatos

16-23 Mild Explocivity, Persistent Homology and Cryptocurrencies' Bubbles: An Empirical Exercise, Stelios Arvanitis and Michalis Detsis

17-23 A network and machine learning approach to detect Value Added Tax fraud, Angelos Alexopoulos, Petros Dellaportas, Stanley Gyoshev, Christos Kotsogiannis, Sofia C. Olhede, Trifon Pavkov

18-23 Time Varying Three Pass Regression Filter, Yiannis Dendramis, George Kapetanios, Massimiliano Marcellino

19-23 From debt arithmetic to fiscal sustainability and fiscal rules: Taking stock, George Economides, Natasha Miouli and Apostolis Philippopoulos

20-23 Stochastic Arbitrage Opportunities: Set Estimation and Statistical Testing, Stelios Arvanitis and Thierry Post

21-23 Behavioral Personae, Stochastic Dominance, and the Cryptocurrency Market, Stelios Arvanitis, Nikolas Topaloglou, and Georgios Tsomidis

22-23 Block Empirical Likelihood Inference for Stochastic Bounding: Large Deviations Asymptotics Under *m*-Dependence, Stelios Arvanitis and Nikolas Topaloglou

23-23 **A Consolidation of the Neoclassical Macroeconomic Competitive General Equilibrium Theory via Keynesianism (Part 1 and Part 2), Angelos Angelopoulos**
24-23 **Limit Theory for Martingale Transforms with Heavy-Tailed Noise, Stelios Arvanitis and Alexandros Louka**

2024

01-24 **Market Timing & Predictive Complexity, Stelios Arvanitis, Foteini Kyriazi, Dimitrios Thomakos**
02-24 **Multi-Objective Frequentistic Model Averaging with an Application to Economic Growth, Stelios Arvanitis, Mehmet Pinar, Thanasis Stengos, Nikolas Topaloglou**
03-24 **State dependent fiscal multipliers in a Small Open Economy, Xiaoshan Chen, Jilei Huang, Petros Varthalitis**
04-24 **Public debt consolidation: Aggregate and distributional implications in a small open economy of the Euro Area, Eleftherios-Theodoros Roumpanis**
05-24 **Intangible investment during the Global Financial Crisis in the EU, Vassiliki Dimakopoulou, Stelios Sakkas and Petros Varthalitis**
06-24 **Time will tell! Towards the construction of instantaneous indicators of different agent-types, Iordanis Kalaitzoglou, Stelios Arvanitis**
07-24 **Norm Constrained Empirical Portfolio Optimization with Stochastic Dominance: Robust Optimization Non-Asymptotics, Stelios Arvanitis**
08-24 **Asymptotics of a QLR-type test for optimal predictive ability, Stelios Arvanitis**
09-24 **Strongly Equitable General Equilibrium Allocations, Angelos Angelopoulos**
10-24 **The Greek macroeconomy: A note on the current situation and future outlook, Apostolis Philippopoulos**
11-24 **Evolution of Greek Tax System, A Survey of Legislated Tax Changes from 1974 to 2018, Panagiotis Asimakopoulos**
12-24 **Macroeconomic Impact of Tax Changes, The case of Greece from 1974 to 2018, Panagiotis Asimakopoulos**
13-24 **`Pareto, Edgeworth, Walras, Shapley' Equivalence in a Small Economy, Angelos Angelopoulos**
14-24 **Stimulating long-term growth and welfare in the U.S, James Malley and Apostolis Philippopoulos**
15-24 **Distributionally Conservative Stochastic Dominance via Subsampling, Stelios Arvanitis**

Department of Economics
Athens University of Economics and Business

The Department is the oldest Department of Economics in Greece with a pioneering role in organising postgraduate studies in Economics since 1978. Its priority has always been to bring together highly qualified academics and top quality students. Faculty members specialize in a wide range of topics in economics, with teaching and research experience in world-class universities and publications in top academic journals.

The Department constantly strives to maintain its high level of research and teaching standards. It covers a wide range of economic studies in micro-and macroeconomic analysis, banking and finance, public and monetary economics, international and rural economics, labour economics, industrial organization and strategy, economics of the environment and natural resources, economic history and relevant quantitative tools of mathematics, statistics and econometrics.

Its undergraduate program attracts high quality students who, after successful completion of their studies, have excellent prospects for employment in the private and public sector, including areas such as business, banking, finance and advisory services. Also, graduates of the program have solid foundations in economics and related tools and are regularly admitted to top graduate programs internationally. Three specializations are offered:1. Economic Theory and Policy, 2. Business Economics and Finance and 3. International and European Economics. The postgraduate programs of the Department (M.Sc and Ph.D) are highly regarded and attract a large number of quality candidates every year.

For more information:

https://www.dept.aueb.gr/en/econ/