Fertility Choices, Human Capital Accumulation, and Endogenous Volatility*

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Abstract

In a three-period overlapping generations model, I show that different combinations of preference and technological parameters can lead to different patterns on the joint evolution of human capital and (endogenous) fertility choices. These patterns may include threshold effects and multiple equilibria as well as endogenous fluctuations. In the latter case, fertility is procyclical. Contrary to existing analyses, endogenous economic fluctuations emerge only when the substitution effects (rather than the income effects) dominate. I also show that the elasticity of intertemporal substitution may be an additional factor determining whether the economy can sustain a positive growth rate in the long-run.

JEL classification: J13, O41

Keywords: Fertility, Human capital, Cycles

* I would like to thank María Gil-Moltó and Ludovic Renou for helpful discussions.

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1 Introduction

For the most part of the last two centuries, the process of economic growth in currently developed countries has been accompanied by some salient demographic trends. In particular, the demographic structures in these countries have changed drastically as both fertility and mortality rates have displayed a clear tendency to fall while life expectancy has risen significantly (Dyson and Murphy, 1985; Kirk, 1996; Ehrlich and Lui, 1997). Greater funding opportunities and technological advancements that have supported both medical and pharmaceutical research, better nutrition, improved sanitation, the design and implementation of health and safety rules – these are some of the reasons that can provide a straightforward explanation on why the process of economic development is accompanied by improvements in health conditions that lead to lower mortality and higher life expectancy. Nevertheless, the apparent reduction in birth rates, which constitutes the other important aspect of demographic transition, in not as straightforward to explain. For this reason, a burgeoning literature has been seeking to provide possible theoretical explanations based on models that account for the joint determination of economic and demographic outcomes (e.g., Becker and Barro, 1988; Becker *et al.*, 1990; Tamura, 1996; Dahan and Tsiddon, 1998; Galor and Weil, 2000; Palivos, 2001; Hazan and Berdugo, 2002; Blackburn and Cipriani, 2002; Lagerlöf, 2003; Kalemli-Ozcan, 2003, Currais *et al.*, 2009; Kitaura, 2009).

Despite this strong interest, however, there is still an aspect of demographic transition that has not received enough attention. Specifically, a closer look at the data reveals that, for the greatest part of the period that constitutes the demographic transition, fertility rates have displayed significant variations around their declining trends. In fact, researchers have identified such patterns for the large majority of developed countries (Easterlin, 1987, 2000; Chesnais, 1992; Lee, 1997). Boldrin *et al.* (2005) offer a more revealing discussion on this issue. They focus on data from the United States and many European countries from the beginning of the 20th century onwards and, in particular, the Baby Bust-Baby Boom-Baby Bust episodes that occurred during that period. Measuring deviations of the Total Fertility Rate and Total Factor Productivity, they show that the periods during which both are either above or below their respective trends are largely coincidental. Thus, they argue that fertility rates are procyclical. If anything, this observation reveals that fluctuations in birth rates are (to a large extent) inherently linked to economic conditions.

The first theoretical model to show the possibility of fertility oscillations (permanent or damped) is that of Kemp and Kondo (1986). They use an overlapping generations model with bequests and derive conditions under which both the economy's capital stock and the fertility rate fluctuate at opposite directions. Benhabib and Nishimura (1989) use the framework of Becker and Barro (1988) in which they generalise the function that determines the relative importance of children's well-being for parent's utility. Their results show that population growth and income per capita can be positively related; they may also display cyclical patterns. Strulik (1999) incorporates life expectancy in a model of optimal saving and fertility decisions. Assuming that life expectancy is positively related to income per capita, he generates rich dynamics for capital accumulation and population growth – dynamics that lead to multiple equilibria and, under circumstances, endogenous cycles in fertility and income per capita. Feichtinger and Dockner (1990) are able to derive endogenous fertility fluctuations by incorporating habit formation in consumption and assuming that the births are an increasing function of the difference between current consumption and a weighted average of past consumption levels. Azariadis (1993) uses a dynamic model where reproductive agents live for one period and production utilises labour and land. He shows conditions under which land usage and population growth admit periodic equilibria. Finally, Jones and Schoonbroodt (2007) deviate from the aforementioned analyses in that they seek to explain fluctuations in fertility as the result of optimal decisions in the presence of exogenous productivity shocks. Thus, rather than focusing on damped oscillations or periodic equilibria emerging from non-monotonicity in the economy's dynamic behaviour, they build a stochastic variant of the Becker and Barro (1988) framework in which they incorporate temporary productivity shocks. They find that, indeed, fertility can be procyclical; therefore, the fertility rate fluctuates around its trend as a result of exogenous productivity disturbances.

In this paper, I construct a model of economic growth with endogenous fertility decisions and human capital accumulation. My model reveals that different combinations of preference and technological parameters determine whether optimal fertility is either a decreasing or an increasing function of the stock of human capital. In the former case, the dynamics of human capital accumulation are monotonic and fertility declines as the economy grows towards its steady state equilibrium. Furthermore, it is also possible that these dynamics are dictated by threshold effects for which initial conditions determine

whether the economy remains in a poverty trap with declining income and increasing fertility or achieves positive transitory growth coupled with decreasing fertility. In the latter case, however, outcomes may be drastically different. Particularly, the emergence of non-monotonic dynamics leads to (damped or permanent) oscillations, i.e., endogenous volatility. Given the response of fertility to differences in the stock of human capital, these dynamics are translated into volatile fertility rates. Another interesting aspect of my model is that, under certain conditions, the elasticity of intertemporal substitution may be an additional factor determining whether the economy sustains a positive growth rate in the long-run.

As an analysis of endogenous economic volatility, the model presented in this paper should not be viewed as yet another framework in which the elasticity of intertemporal substitution is an important source of endogenous cycles. The reason for this is twofold. Firstly, the relative strength of other structural parameters, in addition to the elasticity of intertemporal substitution, is a crucial determinant of the economy's dynamics and the emergence of fluctuations. Secondly, while some other analyses require dominant income effects – alternatively, an elasticity of intertemporal substitution below one – for periodic equilibria to exist (e.g., Grandmont, 1985; Azariadis and Guesnerie, 1986; Benhabib and Laroque, 1988), in my model such equilibria may exist only if the elasticity of intertemporal substitution is above one; that is, endogenous volatility requires dominant substitution effects.[1]

The remaining paper is organised as follows. In Section 2 I describe the economic environment. Sections 3 and 4 analyse the economy's temporary and dynamic equilibrium respectively. In Section 5 I discuss the cyclical nature of fertility decisions and in Section 6 I modify the model so as to allow for long-run growth. Section 7 concludes.

2 The Economy

Consider an artificial economy in which time takes the form of discrete periods that are indexed by $t = 0,1,...,\infty$. The economy is populated by agents who belong to overlapping generations and have a lifespan of three periods – childhood, young adulthood, and old

[1] Under different settings, Huang and Madden (1996) and Lahiri and Puhakka (1998) have also shown that endogenous cycles can emerge even when income effects do not dominate. Huang and Madden (1996) illustrate that this is possible when the demand for labour is inelastic. Lahiri and Puhakka (1998) achieve this result by adding habit persistence.

adulthood. An agent born in period t is for the most part inactive during childhood: she does not make any decisions by herself but attends some type of basic education (provided costlessly and assimilated effortlessly) that allows her to begin her adulthood equipped with the average stock of human capital available at the beginning of period $t+1$. At the beginning of her young adulthood she is also endowed with a unit of time and an entrepreneurial technology that allows her to transform efficient labour (i.e., raw time augmented by the stock of knowledge and expertise) into units of the economy's homogeneous good. During her youth, she also decides how much to consume, how many children to bear, and how much effort to devote for the accumulation of human capital. Given that the economy's homogeneous good is perishable and non-storable, accumulating human capital is the only way she can transfer real resources towards her old age. In particular, when old, she combines her human capital together with a unit of time and produces units of output by utilising her entrepreneurial technology. She decides how much to consume and, at the end of the period, she passes away naturally.

For an agent born in period t, lifetime utility is given by[2]

$$U^t = \frac{\left[\left(c_{t+1}^t\right)^a \left(n_{t+1}^t\right)^{1-a}\right]^{1-\sigma}}{1-\sigma} + \beta \frac{\left(c_{t+2}^t\right)^{1-\sigma}}{1-\sigma}, \quad a, \beta \in (0,1),\ \sigma > 0, \tag{1}$$

where c_{t+1}^t denotes consumption during youth, c_{t+2}^t denotes consumption during old age and n_{t+1}^t denotes the number of children that the agent will give birth to and raise during her youth. The utility function presented in (1) has the following characteristics. Firstly, the elasticity of intertemporal substitution between activities that offer well-being in different periods of a person's lifetime may have a value that differs from one. Secondly, I have used a flexible parameterisation to indicate the relative importance of the components that determine well-being during young adulthood. As we shall see later, both these ideas will have significant implications for the outcomes that transpire in equilibrium.

During her youth, the agent decides how to divide her time between the production of output, the accumulation of human capital and the rearing of her children. I assume that raising a child requires $q > 0$ units of time. Therefore, her consumption during youth is determined by

[2] I employ the convention of using subscripts to indicate the period during which an activity takes place and superscripts to indicate the birth date of the agent that undertakes this activity.

$$c_{t+1}^t = (1 - e_{t+1}^t - qn_{t+1}^t)H_{t+1}, \tag{2}$$

where e_{t+1}^t denotes the fraction of her time spent on accumulating human capital and H_{t+1} is the average stock of human capital in the economy available at the beginning of period $t+1$. Recall that this is the stock of human capital available to the agent at the beginning of her young adulthood.

The agent combines her existing stock of human capital together with the time she devotes towards activities that augment her knowledge and expertise so as to generate the stock of human capital that will be available during her old age. Denoting the latter by h_{t+2}^t, it evolves according to

$$h_{t+2}^t = \varphi e_{t+1}^t H_{t+1}^{\psi}, \quad \varphi > 0, \ \psi \in (0,1). \tag{3}$$

Later, it will become clear that ψ is another parameter with significant implications for the model's main results.

Given that the agent's only reproductive period is her young adulthood, in period $t+2$ she combines the whole unit of time together with her stock of human capital so as to produce output. She uses the income received from this activity to satisfy her consumption needs when old. Hence, her consumption during old adulthood is dictated by

$$c_{t+2}^t = h_{t+2}^t. \tag{4}$$

The previous analysis constitutes the analytical description of the economic environment. Thus, the model summarised through equations (1)-(4) can be used to derive the economy's temporary and dynamic equilibrium and analyse their characteristics. This is a task undertaken in the following Sections.

3 The Temporary Equilibrium

The temporary equilibrium can be described through

Definition 1. *The temporary equilibrium of the economy is a set of quantities* $\{c_{t+1}^t, c_{t+2}^t, n_{t+1}^t, e_{t+1}^t, H_{t+1}, h_{t+2}^t\}$ *such that:*

(i) Given H_{t+1}*, the quantities* c_{t+1}^t*,* c_{t+2}^t*,* e_{t+1}^t*,* n_{t+1}^t *and* h_{t+2}^t *solve the optimisation problem of an agent born in* t*;*

(ii) $h^t_{t+j} = H_{t+j}$ *for* $j = 1, 2, \ldots$.

It is straightforward to establish that the model generates interior equilibria for the variables that comprise the agent's set of choices. For this reason, we can substitute (2), (3) and (4) in (1) in order to express the problem as

$$\left(e^t_{t+1}, n^t_{t+1}\right) = \arg\max\left\{\frac{([(1-e^t_{t+1}-qn^t_{t+1})H_{t+1}]^a (n^t_{t+1})^{1-a})^{1-\sigma}}{1-\sigma} + \beta\frac{(\varphi e^t_{t+1} H^{\psi}_{t+1})^{1-\sigma}}{1-\sigma}\right\}, \tag{5}$$

subject to

$$0 \le e^t_{t+1} \le 1 \text{ and } 0 \le n^t_{t+1} \le 1 .$$

After some tedious but straightforward algebra, we can find the solution for e^t_{t+1} as

$$e^t_{t+1} = \frac{\omega H^{\delta}_{t+1}}{1+\omega H^{\delta}_{t+1}} = \varepsilon(H_{t+1}), \tag{6}$$

where

$$\omega = \beta^{\frac{1}{\sigma}}\left[\frac{\varphi q^{1-a}}{a^a (1-a)^{1-a}}\right]^{\frac{1-\sigma}{\sigma}} \text{ and } \delta = \frac{1-\sigma}{\sigma}(\psi - a). \tag{7}$$

The optimal fertility rate, n^t_{t+1}, is equal to

$$n^t_{t+1} = \frac{1-a}{q}[1-\varepsilon(H_{t+1})] = \nu(H_{t+1}). \tag{8}$$

It is also instructive to write down the solution for the time that the agent devotes for the production of output during her youth, i.e., $1-e^t_{t+1}-qn^t_{t+1}$. Using the results in (6) and (8), this is found to be

$$1-e^t_{t+1}-qn^t_{t+1} = a[1-\varepsilon(H_{t+1})] = l(H_{t+1}). \tag{9}$$

The solutions given in equations (6), (8) and (9) allow us to clarify some previous remarks on the importance of the parameters σ, a and ψ. In order to formalise the argument, let us begin by setting $\sigma = 1$. In this case, the composite parameter terms in (7) become

$$\omega = \beta \text{ and } \delta = 0. \tag{10}$$

Therefore, the solutions in (6), (8) and (9) are reduced to

$$e^t_{t+1} = \frac{\beta}{1+\beta} = \bar{\varepsilon}, \tag{11}$$

$$n_{t+1}^{t} = \frac{1-a}{q(1+\beta)} = \overline{v}\,, \tag{12}$$

and

$$1 - e_{t+1}^{t} - qn_{t+1}^{t} = \frac{a}{1+\beta} = \overline{l}\,. \tag{13}$$

It is obvious that, as long as $\sigma = 1$, the optimal allocation of time during youth is invariant to the existing stock of human capital. The reason for this outcome is as follows. The stock of human capital generates substitution and income effects through its presence in the technology that determines the young adult's output (see equation (2)) and in the technology that determines the accumulation of human capital (see equation (3)) which, in turn, dictates the amount of output at the disposal of the agent during her old age (see equation (4)). If the elasticity of intertemporal substitution is restricted to be equal to one, then the magnitude of these effects is such that they cancel each other out.

Nevertheless, the elasticity of intertemporal substitution is not the only important factor in the determination of a young adult's optimal allocation of time. As mentioned previously, the parameters ψ and a are also crucial in this respect. This argument can be clarified through

Proposition 1. *Consider* $\sigma \neq 1$. *The optimal allocation of time is such that:*

(i) *If* $\sigma \in (0,1)$ *then* $\begin{cases} \varepsilon'(H_{t+1}) > 0,\ l'(H_{t+1}) < 0,\ v'(H_{t+1}) < 0 \quad \textit{for} \quad \psi > a \\ \varepsilon'(H_{t+1}) < 0,\ l'(H_{t+1}) > 0,\ v'(H_{t+1}) > 0 \quad \textit{for} \quad \psi < a \end{cases}$;

(ii) *If* $\sigma > 1$ *then* $\begin{cases} \varepsilon'(H_{t+1}) < 0,\ l'(H_{t+1}) > 0,\ v'(H_{t+1}) > 0 \quad \textit{for} \quad \psi > a \\ \varepsilon'(H_{t+1}) > 0,\ l'(H_{t+1}) < 0,\ v'(H_{t+1}) < 0 \quad \textit{for} \quad \psi < a \end{cases}$.

Proof. Substitute (7) in (6) and use the resulting expression in (8) and (9). Subsequently, calculate the derivatives of $\varepsilon(H_{t+1})$, $l(H_{t+1})$ and $v(H_{t+1})$ with respect to H_{t+1}. □

Let us try to understand the intuition behind these results by considering the impact of a higher human capital stock. Naturally, this implies that the marginal utility cost of devoting time to activities that increase human capital (i.e., the opportunity cost of not producing output during youth) is higher. This is because the higher stock of human capital increases the amount of income received (and, correspondingly, the amount of goods that can be purchased and consumed) for every unit of labour time devoted during the person's youth. The substitution effect induces the agent to increase labour time at the expense of the time she spends accumulating human capital. Given that the number of children that the agent rears is effectively a normal good, the optimal response will be to increase her fertility rate. Nevertheless, there is an income effect as well. Given the convexity of preferences, the agent will optimally wish to smooth her consumption profile over the lifetime. The only way she can achieve this is by accumulating human capital – doing so will increase the resources that she can produce during her old adulthood and, therefore, allow her to consume more during this later stage of her lifespan. This effect will induce her to reduce the time she spends producing output and rearing children during her young adulthood.

There is a second set of substitution and income effects, however. These effects relate to the impact of the current human capital stock on the formation of human capital – consequently, the amount of income received during old age and, therefore, the marginal utility benefit from old age consumption. The substitution effect will induce the agent to spend more time accumulating human capital during her youth, at the expense of the time she spends producing output and raising children. However, there is an income effect related to the fact that the agent wants to smooth her consumption profile. In this case, she can achieve this by increasing her income and, therefore, consumption during youth. Optimally, she will reduce the time spent on the accumulation of human capital and she will increase the time she devotes to the production of output. Since the number of children raised is a normal good, her fertility rate will increase as well.

The elasticity of intertemporal substitution determines whether income or substitution effects dominate. Given the previous discussion, however, we have two sets of such effects that work in the opposite direction. So what determines the ultimate outcome? As summarised in Proposition 1, for given values of σ, the ultimate outcome will be determined by the parameters ψ and a (in particular, the sign of the difference $\psi - a$). We can clarify the intuition as follows. Suppose that $\sigma \in (0,1)$ so that the substitution effects

dominate the income effects. Since the substitution effects are conflicting, the equilibrium outcome will be determined by the relative strengths of ψ and a. Given an increase in the stock of human capital, if $\psi > a$ $(\psi < a)$ the marginal utility benefit from old age consumption is stronger (weaker) compared to the marginal utility benefit from the consumption during young adulthood. Correspondingly, the agent will find optimal to increase (decrease) the time she spends accumulating human capital and, at the same time, reduce (increase) her effort towards other activities – mainly, childrearing and production – during her youth. Now, suppose that $\sigma > 1$ so that the income effects dominate the substitution effects. Again the income effects are conflicting, therefore the equilibrium outcome will be determined by the relative strengths of ψ and a. For a higher stock of human capital, if $\psi > a$ $(\psi < a)$ the marginal utility benefit from consuming when young is stronger (weaker) compared to the marginal utility benefit from consuming when old. As a result, the young adult will find optimal to reduce (increase) the time she spends accumulating human capital. Furthermore, she will optimally increase (reduce) the number of children she bears because increased (reduced) production endows her with more (fewer) resources during her young adulthood.

The qualitative nature of these effects reveals that they may have significant repercussions for the dynamics of human capital accumulation. Consequently, equation (8) reveals that there may also be important implications for the dynamics of fertility. These issues are formally analysed and discussed in the subsequent Section.

4 The Dynamic Equilibrium

In this model, I have assumed that agents within an age-group are identical. Hence, given Definition 1, it is $h^{t}_{t+2} = H_{t+2}$. Substituting this, together with (6), in (3) yields

$$H_{t+2} = \varphi \frac{\omega H_{t+1}^{\psi+\delta}}{1+\omega H_{t+1}^{\delta}} = F(H_{t+1}). \tag{14}$$

This first-order difference equation describes the dynamics of human capital accumulation. By equation (8), this expression will dictate the dynamics of the fertility rate. Thus, the economy's dynamic equilibrium is described in

Definition 2. *The dynamic equilibrium is a sequence of temporary equilibria that satisfy*

(i) $H_{t+2} = F(H_{t+1})$;

(ii) $n^t_{t+1} = v(H_{t+1})$.

Earlier, we identified the fact that the elasticity of intertemporal substitution is a potential source of rich economic and demographic effects, as long as it deviates from the value of one. For this reason, it is instructive to examine the dynamic outcomes that transpire in the baseline case where $\sigma = 1$. I summarise these outcomes in

Proposition 2. *Suppose that* $\sigma = 1$. *There is a unique asymptotically stable steady state* $\bar{H} > 0$. *In the transition to this stationary equilibrium, the fertility rate remains constant, i.e.,* $n^t_{t+1} = \bar{v} \quad \forall t$.

Proof. See the Appendix. □

The human capital dynamics associated with Proposition 1 are illustrated in Figure 1. As the economy converges to its steady-state equilibrium, the fertility rate remains constant because, for reasons explained in the preceding Section, the stock of human capital does not impinge on a young adult's decisions. For the subsequent parts of the analysis, the assumption of a unit elasticity of intertemporal substitution is relaxed.

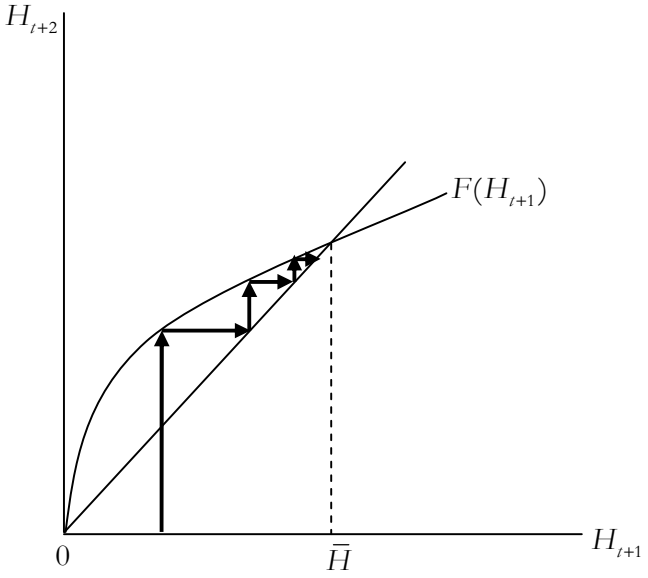

Figure 1

4.1 The Case with $\sigma \in (0,1)$

When the CRRA coefficient (σ) is positive but below one, the elasticity of intertemporal substitution $(1/\sigma)$ takes values above one. This implies that substitution effects dominate income effects and, in terms of our model, leads to the outcomes summarised in the first part of Proposition 1. As we discussed earlier, the qualitative nature of these outcomes depends on the strength of the parameters determining the formation of human capital (ψ) and the relative importance of consumption for a young adult's well-being (a). We can begin the analysis of the economy's dynamics with

Proposition 3. *Suppose that* $\psi > a$.

(i) *If* $\delta + \psi < 1 \Rightarrow a + \dfrac{\sigma}{1-\sigma} > \psi > a$ *then there is a unique asymptotically stable steady state* $H^* > 0$. *In the transition to this stationary equilibrium, the fertility rate declines towards its long-run equilibrium* $v(H^*)$*;*

(ii) *If* $\delta + \psi > 1 \Rightarrow \psi > a + \dfrac{\sigma}{1-\sigma}$ *then there are two asymptotically stable steady states,* $H_1^* = 0$ *and* $H_3^* > 0$*, separated by an unstable steady state* $H_2^* \in (0, H_3^*)$. *For an initial condition below* H_2^**, the stock of human capital declines towards* $H_1^* = 0$ *while the fertility rate increases towards* $v(0)$. *For an initial condition above* H_2^**, the stock of human capital increases towards* H_3^* *while the fertility rate declines towards* $v(H_3^*)$.

Proof. See the Appendix. □

The possible outcomes summarised in Proposition 3 are illustrated in Figures 2 and 3. Recall that when $0 < \sigma < 1$ and $\psi > a$, young adults respond to a higher human capital stock by increasing the time they spend on activities that promote human capital formation at the expense of output production and childrearing. According to equation (3), this result implies that there is a complementarity between the existing human capital stock and human capital investment. When $a + \sigma/(1-\sigma) > \psi$, this complementarity is not strong enough and

therefore, the return to human capital investment is still high when the existing human capital stock is relatively low. As a result, the rate of human capital formation guarantees a unique interior equilibrium. Given that the higher stock of human capital induces agents to reduce the number of children they give birth to, the fertility rate declines as the economy grows towards its steady state. When $\psi > a + \sigma/(1-\sigma)$, the equilibrium outcomes become richer. Now, the complementarity between the human capital stock and human capital investment is strong enough to ensure that, for any $H_{t+1} < H_2^*$, the return to human capital investment is so low that the rate of human capital accumulation is negative. Effectively, H_2^* emerges as an endogenous threshold that determines long-term prospects according to initial conditions. On the one hand, an economy that is endowed with human capital below H_2^* will fall into a poverty trap: the growth rate is negative and the continuously declining human capital stock will lead to increasing fertility rates over time. On the other hand, an economy that is endowed with human capital above H_2^* will grow at positive rates as it converges to its long-run equilibrium: as the stock of human capital increases, the fertility rate falls.[3]

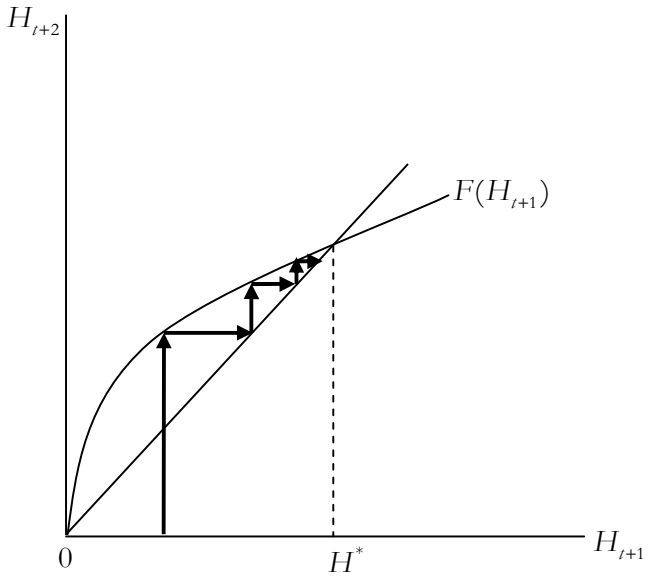

Figure 2

[3] A necessary condition for the existence of interior equilibria in this case is $F(X) > X$ where $X = [(\delta + \psi - 1)/(1-\psi)\omega]^{1/\delta}$. More details are provided in the Appendix.

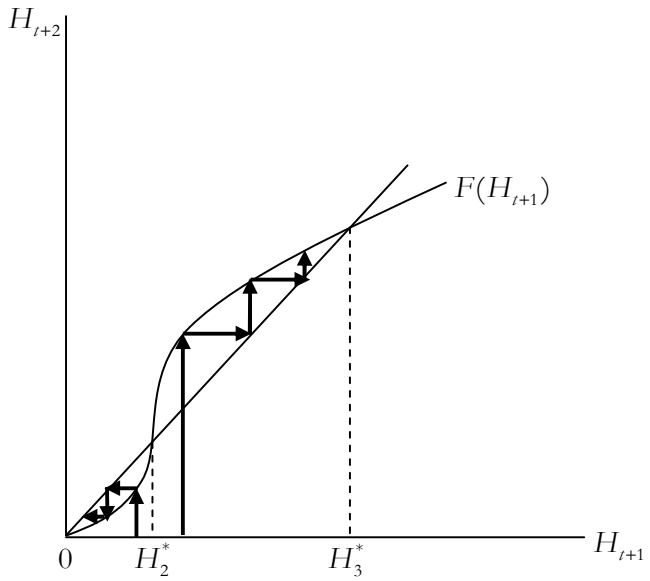

Figure 3

So far, the analysis and discussion have focused on cases where $\psi > a$. Nevertheless, there are parameter configurations for which $\psi < a$. What are the implications for the economy's long-term prospects in this case? One possible outcome is summarised in

Proposition 4. *Suppose that* $\delta + \psi > 0 \Rightarrow a > \psi > (1-\sigma)a$. *There is a unique asymptotically stable steady state* $\tilde{H} > 0$. *In the transition to this stationary equilibrium, the fertility rate increases towards its long-run equilibrium* $v(\tilde{H})$.

Proof. See the Appendix. □

Recall that, as long as $\psi < a$, a higher stock of human capital induces young adults to spend more time producing output and raising children – at the same time, they devote less time for the accumulation of human capital. In principle, this implies that when the stock of human capital is high enough, the return to human capital investment may become negative. Nonetheless, when $a > \psi > (1-\sigma)a$ this situation does not emerge. The return to human

capital investment is still sufficient enough to ensure that the economy will grow monotonically towards its long-run equilibrium. The difference with previous scenarios is that the fertility rate actually increases during the transition period because, for $\psi < a$, young adults respond to the higher human capital stock by increasing the time they spend towards income bearing activities and, consequently, childrearing. This scenario is illustrated in Figure 4.

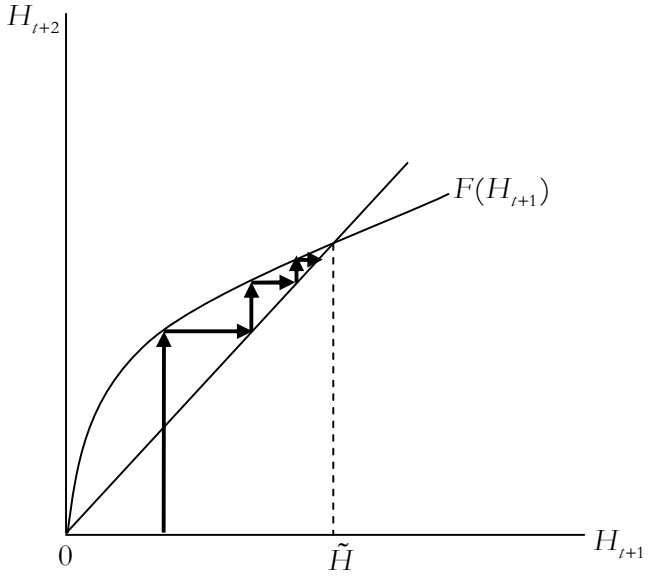

Figure 4

The most interesting dynamics emerge with the case where $\psi < (1-\sigma)a$. This is the scenario where endogenous volatility in economic activity and fertility rates becomes a possible outcome. I summarise this scenario in

Proposition 5. *Suppose that* $\delta + \psi < 0 \Rightarrow \psi < (1-\sigma)a$. *There is a unique interior steady state* $\hat{H} > 0$ *such that:*

(i) *If* $\left|F'(\hat{H})\right| < 1$ *then* $\hat{H}$ *is asymptotically stable. The transition to this stationary equilibrium may not necessarily be monotonic. Instead, human capital may converge to* $\hat{H}$ *through damped oscillations, i.e.,* $(H_{t+j} - \hat{H})(H_{t+j+1} - \hat{H}) < 0$ *and* $\lim_{t\to\infty}(H_{t+j} - \hat{H})(H_{t+j+1} - \hat{H}) = 0$

for $j = 1, 2, \ldots$. *When this happens, the fertility rate also converges to its long-run equilibrium* $v(\hat{H})$ *through damped oscillations, i.e.,* $[v(H_{t+j}) - v(\hat{H})][v(H_{t+j+1}) - v(\hat{H})] < 0$ *and* $\lim_{t\to\infty}[v(H_{t+j}) - v(\hat{H})][v(H_{t+j+1}) - v(\hat{H})] = 0$ *for* $j = 1, 2, \ldots$.

(ii) *If* $F'(\hat{H}) < -1$ *then* $\hat{H}$ *is unstable. Oscillations in the dynamics of human capital are permanent, i.e., there may be two or more periodic equilibria. If there are two periodic equilibria* $\breve{H}_1$ *and* $\breve{H}_2$, *they satisfy* $0 < \breve{H}_1 < \hat{H} < \breve{H}_2$ *and* $(\breve{H}_1 - \hat{H})(\breve{H}_2 - \hat{H}) < 0$. *Correspondingly, the fertility rate will oscillate permanently as well. That is,* $[v(\breve{H}_1) - v(\hat{H})][v(\breve{H}_2) - v(\hat{H})] < 0$.

Proof. See the Appendix. □

The dynamics described in these cases are illustrated in Figures 5 and 6. At relatively high values for the human capital stock, the slope of the transition equation in (14) may actually change sign and become negative. If the fixed point generated by (14) lies on the downward sloping part of the dynamics, convergence to the steady-state may be cyclical rather than monotonic. There may even be convergence to a stable cycle – human capital may fluctuate permanently around its fixed point. The intuition for this result is the following. Suppose that the human capital stock is low. Young adults will respond by devoting more time to the accumulation of human capital and less time producing output. With lower income, they choose to raise fewer children as well. Next period however, the available stock of human capital will be relatively high as a result of the previous generation's effort. This will induce young adults to produce more output and raise more children at the expense of the time they spend accumulating human capital. The latter effect implies a lower endowment of human capital for the subsequent generation of young adults. As a result, they will decide to invest more time to the accumulation of human capital, at the expense of output production and childrearing, and so on.[4]

[4] In terms of a numerical example, suppose that $a = 0.8$, $\varphi = 1.5$, $\sigma = 0.1$, $\psi = 0.3$, $q = 0.01$ and $\beta = 0.8$. Then $\hat{H} = 0.606848$ and $F'(\hat{H}) = -2.08516$. In this case, human capital and fertility rates fluctuate around their respective fixed points.

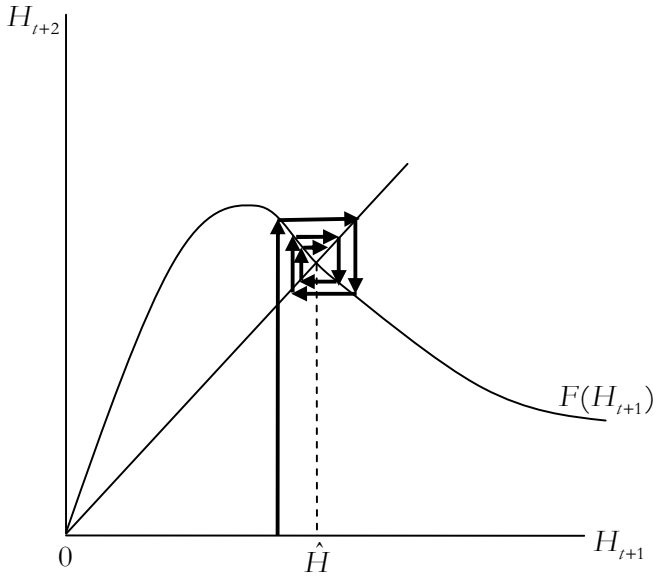

Figure 5

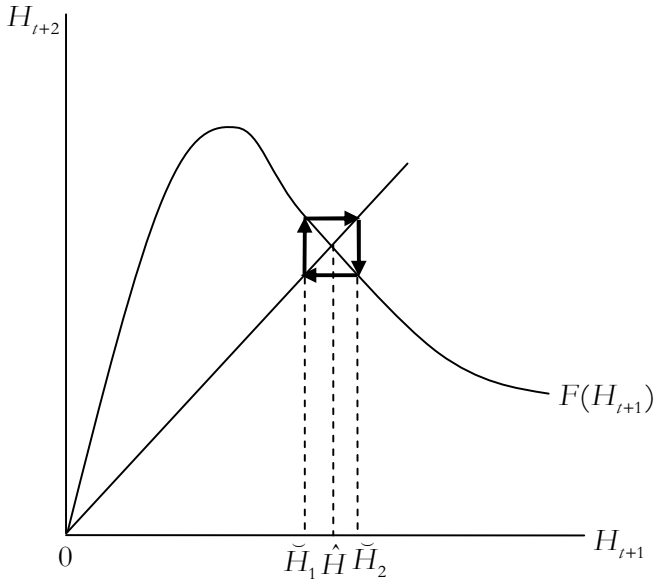

Figure 6

4.2 The Case with $\sigma > 1$

In this part we will identify the equilibrium outcomes that transpire when the elasticity of intertemporal substitution $(1/\sigma)$ takes values below one – the case where income effects dominate substitution effects. In terms of the optimal allocation of time by young adults, the results are summarised in the second part of Proposition 1. In terms of dynamics, however, the results are not as rich as those derived in the case where $1/\sigma > 1$. We can clarify this argument through

Proposition 6. *There is a unique asymptotically stable steady state* $H^+ > 0$. *In the transition to this stationary equilibrium, the fertility rate declines (increases) towards its long-run equilibrium* $v(H^+)$ *as long as* $a > \psi$ $(a < \psi)$.

Proof. From equation (7), we can see that δ can be written as $\delta = \frac{\sigma - 1}{\sigma}(a - \psi)$. Notice that, for $\sigma > 1$ it is always true that $0 < \delta + \psi < 1$. Thus, Proposition 6 follows from the results in Propositions 1, 3 (part (i)) and 4. □

A straightforward comparison of the outcomes analysed in this part as opposed to those analysed in the previous one, allow us to derive an important implication. This implication relates to the importance of the elasticity of intertemporal substitution and takes the form of

Corollary 1. *As long as* $\psi, a \in (0,1)$*, threshold effects and/or endogenous volatility emerge if and only if* $\sigma \in (0,1)$.

5 The Cyclicality of Fertility Choices

As I mentioned in a previous part of the paper, empirical evidence suggests that variations in fertility appear to be procyclical. Of course, this does not necessarily imply that fertility choices display the same high frequency fluctuations that we observe for such macroeconomic variables as investment and output. Nonetheless, the comparison of waves in economic activity and birth rates reveal that periods during which both are either above or below their respective trends are remarkably coincidental (Boldrin *et al.*, 2005).

The current model is able to capture this stylised fact. This can be illustrated as follows. In every period there are two cohorts of agents producing output – young adults and old adults. Given the model's assumptions, each young adult earns income according to

$$y_{young}(H_{t+1}) = l(H_{t+1})H_{t+1}, \tag{15}$$

whereas each old adult's income is equal to

$$y_{old}(H_{t+1}) = H_{t+1}. \tag{16}$$

The result in Proposition 1 reveals that, as long as parameter values are conducive to the emergence of endogenous volatility, it is $l'(H_{t+1}) > 0$ and $v'(H_{t+1}) > 0$. Therefore, we have $y'_{young}(H_{t+1}) > 0$ and $y'_{old}(H_{t+1}) > 0$.

Now, consider some period T and suppose that the model generates endogenous fluctuations, either through damped oscillations or periodic equilibria – say a 2-period cycle. It is straightforward to see that

$$v(H_T) - v(\hat{H}) \begin{cases} > 0 & \text{when} \quad y_\eta(H_T) - y_\eta(\hat{H}) > 0 \\ & \qquad\qquad\qquad\qquad , \quad \text{for } \eta = \{young, old\}. \\ < 0 & \text{when} \quad y_\eta(H_T) - y_\eta(\hat{H}) < 0 \end{cases} \tag{17}$$

The main implication from equation (17) can be summarised in

Corollary 2. *When fertility displays endogenous fluctuations, then these fluctuations are procyclical in the sense that fertility is above (below) its trend as long as output is above (below) its trend.*

6 The Elasticity of Intertemporal Substitution and Growth

So far, my analysis has been based on a human capital accumulation technology that, due to ψ being lower than one, cannot sustain an equilibrium with positive growth in the long-run. Henceforth, this assumption is relaxed and I set $\psi = 1$. In this case, equation (14) is written as

$$H_{t+2} = \varphi \frac{\omega H_{t+1}^{1+\delta}}{1 + \omega H_{t+1}^{\delta}} = F(H_{t+1}), \tag{18}$$

where, given (7), the composite parameter term δ becomes

$$\delta = \frac{1-\sigma}{\sigma}(1-a). \tag{19}$$

We can use (18) to write an expression for the growth rate. That is

$$g_{t+2} = \frac{H_{t+2}}{H_{t+1}} - 1 = \varphi \frac{\omega H_{t+1}^{\delta}}{1+\omega H_{t+1}^{\delta}} - 1. \tag{20}$$

Obviously, given $0 \leq e_{t+1}^{t} \leq 1$, the additional restriction $\varphi > 1$ is required so as to render positive growth possible. With these results in mind, the analysis of the model's dynamics allows us to derive

Proposition 7. *Suppose that* $\sigma \neq 1$.

(i) *If* $\sigma \in (0,1)$ *then there is an asymptotically stable steady state* $\widehat{H}_1 = 0$ *and an unstable steady state* $\widehat{H}_2$. *For an initial condition below* $\widehat{H}_2$, *the stock of human capital declines towards* $\widehat{H}_1 = 0$ *while the fertility rate increases towards* $v(0)$. *For an initial condition above* $\widehat{H}_2$, *the dynamics of human capital accumulation converge to a long-run growth equilibrium where* $\lim_{t\to\infty} g_{t+2} = \widehat{g} = \varphi - 1 > 0$. *As the stock of human capital grows continuously, fertility declines.*

(ii) *If* $\sigma > 1$ *then there is a unique asymptotically stable steady state* $\widehat{H} > 0$. *In the long-run, the economy will converge to an equilibrium with zero economic growth, i.e.,* $\lim_{t\to\infty} g_{t+2} = 0$. *In the transition to the stationary equilibrium* $\widehat{H}$, *the fertility rate increases towards its long-run equilibrium* $v(\widehat{H})$.

Proof. See the Appendix. □

The dynamics of human capital are illustrated in Figures 7 and 8. In terms of intuition, the mechanisms are more or less the same to those described in Proposition 3 (part (ii)) and Proposition 4. The difference of course is that positive long-run growth is now an equilibrium outcome. This outcome is not warranted though. Whether it materialises or not depends on the value of the parameter σ. This important implication is emphasised in

Corollary 3. *For an economy whose initial condition exceeds the endogenous threshold* $\widehat{H}_2$*, an elasticity of intertemporal substitution below one is a deterrent to long-run economic growth.*

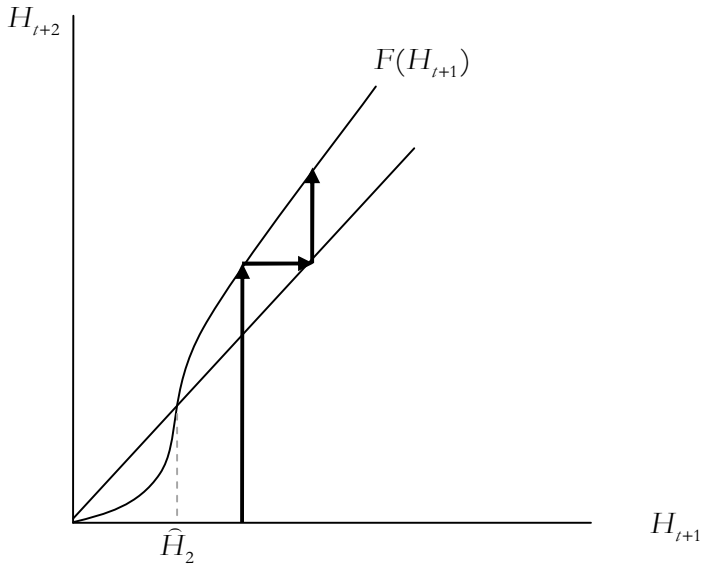

Figure 7

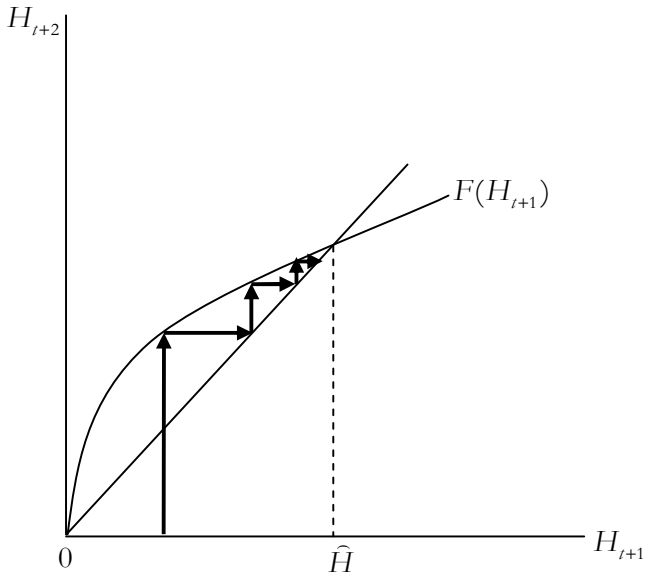

Figure 8

7 Conclusion

What is the underlying link between birth rates and economic activity? Under what conditions can we explain the observed patterns in the evolution of fertility rates and per capita GDP? The purpose of this paper was to shed more light on the fundamental mechanisms that shape the relationship between economic activity and demographic change. Different combinations of technological and preference parameters have been identified as crucial in generating a variety of patterns on the joint evolution of human capital accumulation and fertility rates. These patterns may include threshold effects and multiple equilibria as well as endogenous fluctuations. Such results find support from existing empirical evidence; hence the model offers mechanisms that improve our understanding on the possible driving forces behind salient features of demographic transition and economic growth.

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Appendix

A1 Proof of Proposition 2

Substituting (10) in (14) yields

$$H_{t+2} = \varphi \frac{\beta}{1+\beta} H_{t+1}^{\psi} = F(H_{t+1}). \tag{A1.1}$$

We can see that there are two possible steady state equilibria. One is $H_{t+2} = H_{t+1} = 0$ and the other is $H_{t+2} = H_{t+1} = \bar{H} = [\varphi\beta/(1+\beta)]^{1/(1-\psi)}$. Given (A1.1), the derivative of $F(H_{t+1})$ is

$$F'(H_{t+1}) = \psi\varphi \frac{\beta}{1+\beta} H_{t+1}^{\psi-1}. \tag{A1.2}$$

It is straightforward to examine that $F'(0)=\infty$, $F'(\infty)=0$ and $F'(\bar{H})=\psi\in(0,1)$. Thus, the only asymptotically stable equilibrium is $\bar{H}$. Furthermore, from equation (12) we can see that $n_{t+1}^{t}=\bar{\nu}$. Thus, for $H_{t+1}<\bar{H}$ the stock of human capital grows towards its stationary equilibrium but the fertility rate remains constant.

A2 Proof of Proposition 3

Given (14), we can establish that $F(0)=0$ and $F(\infty)=\infty$. The derivative of $F'(H_{t+1})$ is given by

$$F'(H_{t+1})=\frac{\varphi\omega H_{t+1}^{\delta+\psi-1}}{1+\omega H_{t+1}^{\delta}}\left[\psi+\delta\left(1-\frac{\omega H_{t+1}^{\delta}}{1+\omega H_{t+1}^{\delta}}\right)\right], \tag{A2.1}$$

which is positive for $\psi>a$ Using (A2.1), it is straightforward to establish $F'(\infty)=0$. However, the value of $F'(0)$ depends on different parameter combinations. In particular, it is

$$F'(0)=\begin{cases}0 & \text{if} \quad a+\dfrac{\sigma}{1-\sigma}>\psi>a\\ & \\ \infty & \text{if} \quad \psi>a+\dfrac{\sigma}{1-\sigma}\end{cases}. \tag{A2.2}$$

Therefore, when $\psi>a+\sigma/(1-\sigma)$ $(a<\psi<a+\sigma/(1-\sigma))$ an equilibrium with $H_{t+2}=H_{t+1}=0$ is asymptotically stable (unstable).

Now, define the function

$$M(H_{t+1})=\frac{F(H_{t+1})}{H_{t+1}}=\varphi\frac{\omega H_{t+1}^{\delta+\psi-1}}{1+\omega H_{t+1}^{\delta}}, \tag{A2.3}$$

for which it is straightforward to establish that

$$M(0)=\begin{cases}\infty & \text{if} \quad a+\dfrac{\sigma}{1-\sigma}>\psi>a\\ & \\ 0 & \text{if} \quad \psi>a+\dfrac{\sigma}{1-\sigma}\end{cases} \quad \text{and} \quad M(\infty)=0. \tag{A2.4}$$

The derivative $M'(H_{t+1})$ is equal to

$$M'(H_{t+1}) = \frac{\varphi\omega H_{t+1}^{\delta+\psi-2}}{1+\omega H_{t+1}^{\delta}}\left[\delta\left(1-\frac{\omega H_{t+1}^{\delta}}{1+\omega H_{t+1}^{\delta}}\right)-(1-\psi)\right]. \tag{A2.5}$$

Using (A2.5), we can show that

$$M'(H_{t+1}) < 0, \text{ for } a+\frac{\sigma}{1-\sigma} > \psi, \tag{A2.6}$$

and

$$M'(H_{t+1}) \begin{cases} > 0 & \text{if} \quad H_{t+1} < X \\ & \text{, for } \psi > a+\frac{\sigma}{1-\sigma}, \\ < 0 & \text{if} \quad H_{t+1} > X \end{cases} \tag{A2.7}$$

where

$$X = \left[\frac{\delta+\psi-1}{(1-\psi)\omega}\right]^{\frac{1}{\delta}}. \tag{A2.8}$$

Suppose that $\psi < a+\sigma/(1-\sigma)$. Given (A2.4) and (A2.6), we conclude that there is a unique H^* such that $M(H^*)=1$ which, given (A2.3), corresponds to $H^* = F(H^*)$. Furthermore, $M'(H^*)<0$ is equivalent to $F'(H^*)<1$. Combining these results with $F'(H_{t+1})>0$ and (A2.2), we can conclude that H^* is the unique asymptotically stable steady state. For $H_{t+1} < H^*$ it is $\lim_{t\to\infty} H_{t+1} = H^*$. During the transition, the fertility declines because $v'(H_{t+1})<0$.

Now, suppose that $\psi > a+\sigma/(1-\sigma)$. As long as $M(X)>1$, equations (A2.4) and (A2.7) reveal that there are two equilibria $H_3^* > X > H_2^* > 0$ such that $M(H_2^*) = M(H_3^*) = 1$ and, therefore, $H_2^* = F(H_2^*)$ and $H_3^* = F(H_3^*)$. Given (A2.7), it is $M'(H_2^*)>0 \Leftrightarrow F'(H_2^*)>1$ and $M'(H_3^*)<0 \Leftrightarrow F'(H_3^*)<1$. Combined with $F'(H_{t+1})>0$ and (A2.2), these results reveal that $H_1^*=0$ and H_3^* are asymptotically stable steady state equilibria. On the one hand, if $H_{t+1} < H_2^*$ then $\lim_{t\to\infty} H_{t+1} = 0$ and $\lim_{t\to\infty} v(H_{t+1}) = v(0)$. On the other hand, if $H_{t+1} > H_2^*$ then $\lim_{t\to\infty} H_{t+1} = H_3^*$ and $\lim_{t\to\infty} v(H_{t+1}) = v(H_3^*)$.

A3 Proof of Proposition 4

Consider $a > \psi > (1-\sigma)a$. Combining (A2.2) with (7), we can establish that $F'(H_{t+1}) > 0$, $F(\infty) = \infty$ and $F'(\infty) = 0$. Furthermore, $F(0) = 0$ and $F'(0) = \infty$ which imply that $H_{t+2} = H_{t+1} = 0$ is an unstable steady state. Next, let us use $a > \psi > (1-\sigma)a$ in (A2.3) to establish that

$$M(0) = \infty \quad \text{and} \quad M(\infty) = 0. \tag{A3.1}$$

Furthermore, applying $a > \psi > (1-\sigma)a$ to (A2.5) reveals that $M'(H_{t+1}) < 0$.

All these results allow us to verify that there is a unique $\tilde{H}$ such that $M(\tilde{H}) = 1$ or, alternatively, $\tilde{H} = F(\tilde{H})$. Furthermore, $M'(\tilde{H}) < 0$ is equivalent to $F'(\tilde{H}) < 1$. However, it is $F'(H_{t+1}) > 0$. Thus, $\tilde{H}$ is asymptotically stable and for $H_{t+1} < \tilde{H}$ it is $\lim_{t\to\infty} H_{t+1} = \tilde{H}$. During the transition, the fertility increases because $v'(H_{t+1}) > 0$.

A4 Proof of Proposition 5

If $\psi < (1-\sigma)a$, then we can utilise (A2.1) to establish that $F(0) = F(\infty) = 0$ and

$$F'(H_{t+1}) \begin{cases} > 0 & \text{if} \quad H_{t+1} < \Xi \\ & \\ < 0 & \text{if} \quad H_{t+1} > \Xi \end{cases}, \tag{A4.1}$$

where

$$\Xi = \left[\frac{\omega\psi}{-(\delta+\psi)} \right]^{\frac{1}{-\delta}}. \tag{A4.2}$$

Notice that $\Xi > 0$ because $-(\delta+\psi) > 0$ for $\psi < (1-\sigma)a$. Moreover, $F(0) = 0$ and $F'(0) = \infty$. This verifies that $H_{t+2} = H_{t+1} = 0$ is an unstable steady state.

We can also apply $\psi < (1-\sigma)a$ to (A2.3) and (A2.4) to derive $M(0) = \infty$, $M(\infty) = 0$ and $M'(H_{t+1}) < 0$. These results imply that there is a unique $\hat{H}$ such that $M(\hat{H}) = 1 \Leftrightarrow \hat{H} = F(\hat{H})$. Furthermore, $M'(\hat{H}) < 0$ or, alternatively, $F'(\hat{H}) < 1$. In this case, however, we cannot make any definite conclusions concerning the stability of this equilibrium as we do not yet know whether $\hat{H}$ lies on the downward sloping part of

$F(H_{t+1})$, in which case $F'(\hat{H})<0$. For this reason, we have to examine two different scenarios.

Let us begin with the case where $-1<F'(\hat{H})<0$. As long as this condition is satisfied, $\hat{H}$ is asymptotically stable but convergence towards it may be cyclical rather than monotonic. We can see this by employing a linear approximation of (14) around $\hat{H}$ and solve for some period T. That is,

$$H_T-\hat{H}=[F(\hat{H})]^T(H_0-\hat{H}). \tag{A4.3}$$

Given $F'(\hat{H})<0$, this approximation reveals that for odd (even) values of T, H_T lies above (below) its long-run equilibrium value. Furthermore, equation (8) reveals that these damped oscillations will generate similar oscillations to the fertility rate.

Next, let us consider the case where $F'(\hat{H})<-1$. In this situation, $\hat{H}$ is an unstable steady state. However, because it lies on the downward sloping part of the transition equation, the dynamics of human capital will converge to a stable cycle, i.e., the model will admit periodic equilibria. This result can be proven as follows. As long as $F'(\hat{H})<0$, any $\zeta>\hat{H}$ satisfies $\zeta>F(\zeta)$. Furthermore, given $F(0)=F(\infty)=0$, there must be some $\zeta>\hat{H}$ which is high enough to satisfy $\zeta>F(\Xi)>\Xi$ (recall that $F(\Xi)$ is a global maximum and $F'(0)=\infty$) and $F(\zeta)<\Xi$. However, the latter implies that $F^2(\zeta)<F(\Xi)\Leftrightarrow\zeta>F^2(\zeta)$, where $F^2(\zeta)=F(F(\zeta))$. Therefore, there exists at least one ζ such that $\zeta>F(\zeta)$, $\zeta>F^2(\zeta)$ and therefore, according to Theorem 8.2 in Azariadis (1993), the condition $F'(\hat{H})<-1$ is sufficient for the existence of a 2-period cycle (with periodic equilibria $\breve{H}_1$ and $\breve{H}_2$) such that $0<\breve{H}_1<\hat{H}<\breve{H}_2<\zeta$. Correspondingly, the result in equation (8) reveals that the fertility rate will admit periodic equilibria as well.

A5 Proof of Proposition 7

For $\psi=1$, we can use (A2.3) to write the function $M(H_{t+1})$ as

$$M(H_{t+1})=\frac{F(H_{t+1})}{H_{t+1}}=\varphi\frac{\omega H_{t+1}^{\delta}}{1+\omega H_{t+1}^{\delta}}, \tag{A5.1}$$

while its derivative $M'(H_{t+1})$ is equal to

$$M'(H_{t+1}) = \frac{\varphi\omega H_{t+1}^{\delta-1}}{1+\omega H_{t+1}^{\delta}}\delta\left(1-\frac{\omega H_{t+1}^{\delta}}{1+\omega H_{t+1}^{\delta}}\right). \tag{A5.2}$$

Furthermore, it is $F(0)=0$, $F(\infty)=\infty$ and

$$F'(H_{t+1}) = \frac{\varphi\omega H_{t+1}^{\delta}}{1+\omega H_{t+1}^{\delta}}\left[1+\delta\left(1-\frac{\omega H_{t+1}^{\delta}}{1+\omega H_{t+1}^{\delta}}\right)\right] > 0. \tag{A5.3}$$

Let us begin with the case where $\sigma \in (0,1)$. We can use (A5.1), (A5.2) and (A5.3) to establish that $F'(0)=0$, $F'(\infty)=\varphi>1$, $M(0)=0$, $M(\infty)=\varphi>1$ and $M'(H_{t+1})>0$. Therefore, there is a unique $\widehat{H}_2>0$ such that $M(\widehat{H}_2)=1 \Leftrightarrow F(\widehat{H}_2)=\widehat{H}_2$. Furthermore, it is $M'(\widehat{H}_2)>0$ which means that $F'(\widehat{H}_2)>1$. This analysis reveals that $\widehat{H}_1=0$ is a stable equilibrium while $\widehat{H}_2$ is unstable. If $H_{t+1}<\widehat{H}_2$ then $\lim_{t\to\infty} H_{t+1}=0$ and $\lim_{t\to\infty} v(H_{t+1})=v(0)$. If $H_{t+1}>\widehat{H}_2$ then $\lim_{t\to\infty} H_{t+1}=\infty$ because $H_{t+2}>H_{t+1}$ $\forall t$. Thus, we can use (20) to get $\lim_{t\to\infty}\left(\frac{H_{t+2}}{H_{t+1}}-1\right)=\varphi-1>0$ which, given (8), means that fertility will decline continuously as human capital grows.

Now, let us consider $\sigma>1$. In this case, we have $F'(0)=\varphi>1$, $F'(\infty)=0$, $M(0)=\varphi>1$, $M(\infty)=0$ and $M'(H_{t+1})<0$. Given these, $H_{t+2}=H_{t+1}=0$ is an unstable equilibrium while $\widehat{H}>0$ such that $M(\widehat{H})=1 \Leftrightarrow F(\widehat{H})=\widehat{H}$ exists. It is a stable steady state because $F'(H_{t+1})>0$, and $M'(\widehat{H})<0 \Leftrightarrow F'(\widehat{H})<1$. Therefore, $\lim_{t\to\infty} H_{t+1}=\widehat{H}$ and, by virtue of (8), fertility will be increasing towards $\lim_{t\to\infty} v(H_{t+1})=v(\widehat{H})$.